CONICS AND CUBICS CONNECTED WITH A PLANE CUBIC BY CERTAIN COVARIANT RELATIONS*

BY

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It is to be expected that the systematic introduction of irrational covariants will enrich geometry with curves and surfaces, not previously observed or discussed, allied to any given fundamental system by projective relations. Well known irrationalities also must be expected often to reappear. In 1888 Dr. HILBERT † remarked the existence of two nets of conics covariantly related to a general curve of third order in a plane. Two additional systems, nets of curves of the second class, can easily be defined by an equation closely analogous to Hilbert's. These systems of conics and the four cubics whose polars they are prove to be not entirely unknown hitherto, for their dually equivalent loci and envelopes form the basis of the one-to-one correspondences upon the point-cubic and the line-cubic respectively. That only two of every three such correspondences are found here is due to the domain of rationality that is assumed. By employing the irrationality that occurs in HESSE'S canonical form of the cubic I am able to identify Hilbert's two systems of irrational covariant conics and to exhibit their relation to the other two systems just mentioned. As a consequence it is found possible to give explicitly covariant equations of definition for the two cubics which have the same Hessian and for those which have the same Cayleyan as a given fundamental cubic. These results are here derived by the aid of a canonical form of the cubic containing Hesse's irrationality. The desirable invariantive proofs of these results are given by Professor GORDAN in a paper presented by him to the American Mathematical Society for publication in its Transactions.‡

§ 1. *Hilbert's covariant conics.*

The covariant of second degree in the coefficients of a ternary cubic a or a_x^3 is written symbolically

$$(aa'u)^2 a_x a'_x.$$

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† *Lettre adressée à M. Hermite*, Liouville's Journal, vol. 4, ser. 4, 1888.

‡ P. 9 of this number.

If this is equated to zero it denotes in variables x_1, x_2, x_3, the poloconic of the line

$$u_1x_1 + u_2x_2 + u_3x_3 = 0.$$

Or if u_1, u_2, u_3 be taken as variables it is the tangential equation of the conic polar of a point (x). If now the variables (u) are replaced by symbols of any ternary n-ic A_x^n, we shall say that the equation

$$(aa'A)^2a_xa'_xA_x^{n-2} = 0 \tag{1}$$

denotes the poloconic, polocubic, or polo-n-ic of the curve $A_x^n = 0$ with respect to the cubic $a_x^3 = 0$. HILBERT proposes to determine those conics, if any, which coincide with their poloconics. This requires the determination of a value λ such that

$$(aa'A)^2a_xa'_x \equiv \lambda \cdot A_x^2. \tag{2}$$

It is found that the six values of λ resulting from elimination of the unknown A's coincide by threes. Hence to the two values

$$\lambda = \pm \sqrt{\tfrac{1}{6}S} = \pm \sqrt{\tfrac{1}{6}(aa'a'')(aa'a''')(aa''a''')(a'a''a''')}$$

there correspond two quadric forms each containing linearly three arbitrary parameters. So much HILBERT states. In order to recognize these autopoloconics as known systems it will be convenient to use a canonical form of the fundamental cubic, due to HESSE.*

Referred to an inflexional triangle, the equation of the cubic takes the form

$$a_x^3 = x_1^3 + x_2^3 + x_3^3 + 6mx_1x_2x_3 = 0. \tag{3}$$

All conic polars accordingly have the form:

$$a_y a_x^2 = (y_1x_1^2 + y_2x_2^2 + y_3x_3^2) + 2m(y_1x_2x_3 + y_2x_3x_1 + y_3x_1x_2) = 0. \tag{4}$$

With respect to this triangle, the quantity m may be termed indifferently the parameter of the cubic or the parameter of any one of its conic polars. The defining equation of the autopoloconics may be written with the aid of a bordered Hessian, as follows:

$$2\begin{vmatrix} x_1 & mx_3 & mx_2 & A_1 \\ mx_3 & x_2 & mx_1 & A_2 \\ mx_2 & mx_1 & x_3 & A_3 \\ A_1 & A_2 & A_3 & 0 \end{vmatrix} \equiv \lambda A_x^2. \tag{5}$$

Expanded, this identity gives three pairs of equations of condition like the following,

*Crelle's Journal, vol. 28, pp. 68-96, 1844.

$$(6)\qquad \left.\begin{aligned} 2m^2A_{11} + 4mA_{23} &= \lambda A_{11},\\ 2A_{11} + 4m^2A_{23} &= -2\lambda A_{23}.\end{aligned}\right\}$$

Eliminate λ and write

$$\frac{A_{23}}{A_{11}} = A = \frac{A_{31}}{A_{22}} = \frac{A_{12}}{A_{33}}.$$

There results the simple relation between parameters m and A:

$$(7)\qquad 4mA(m + A) + 1 = 0.$$

This yields two values for A, which we shall distinguish as A and B. The corresponding systems or nets of conics are these:

$$(8)\qquad \left.\begin{aligned} &(y_1x_1^2 + y_2x_2^2 + y_3x_3^2) + 2A\,(y_1x_2x_3 + y_2x_3x_1 + y_3x_1x_2) = 0,\\ &(y_1x_1^2 + y_2x_2^2 + y_3x_3^2) + 2B\,(y_1x_2x_3 + y_2x_3x_1 + y_3x_1x_2) = 0,\end{aligned}\right\}$$

the y's denoting arbitrary quantities. Notice now that these are the nets of conic polars belonging to the two cubics, which we will denote by corresponding symbols:

$$(9)\qquad \left.\begin{aligned} A_x^3 &= x_1^3 + x_2^3 + x_3^3 + 6Ax_1x_2x_3 = 0,\\ B_x^3 &= x_1^3 + x_2^3 + x_3^3 + 6Bx_1x_2x_3 = 0.\end{aligned}\right\}$$

These two cubics, from the form of their equations, are seen to belong to the syzygetic sheaf determined by the fundamental cubic a and its Hessian. We have then this theorem:

To each general cubic curve there correspond two other cubics in its syzygetic sheaf, the conic polars of the latter being all autopoloconics of the former.

Further, since equation (7) is symmetric in the two parameters m and A, we may add:

With respect to each of the latter two cubics the conic polars of the former constitute one net of autopoloconics.

But the two derivative cubics stand to each other in the same reciprocal relation; for from the conditions (7),

$$4mA(m + A) + 1 = 0,\ 4mB(m + B) + 1 = 0,\ A \neq B,$$

there follows the relation

$$(10)\qquad 4AB(A + B) + 1 = 0.$$

Hence we have the theorem:

These three cubics, the fundamental a and the two derived cubics A and B, constitute a closed system, the autopoloconics of each cubic being the conic polars of each of the other two cubics.

Only one more step is needful for the complete recognition of this set of three. Modify the relation (7) between parameters by the factor $(m - A)/mA$; in the result,

$$4(m^2 - A^2) + \left(\frac{1}{A} - \frac{1}{m}\right) = 0,$$

the parameters can be separated, so that by use of (10) we have the equality

$$-\frac{4m^3 - 1}{6m} = -\frac{4A^3 - 1}{6A} = -\frac{4B^3 - 1}{6B} = c. \tag{11}$$

This rational function having the same value for all three parameters is seen on inspection to be the parameter of the Cayleyan curve of third class, (the envelope of lines constituting degenerate conic polars in each net). Since every non-singular cubic class-curve is Cayleyan to three point-cubics, the relation may be re-stated thus:

The three cubics which have a Cayleyan in common are characterized fully by the property that all conic polars of one (and therefore of each one) are autopolo-conics of both the others.

§2. *Two covariant systems of curves of second class.*

The mixed covariant $\Theta = (aa'u)^2 a_x a'_x$, used above as a transformer for forms in variables (x), may be used equally well to transform those with variables (u). The curve $u_a^n = 0$ is transformed into the curve $(aa'u)^2 a_a a'_a u_a^{n-2} = 0$. If the curve is transformed thus into itself, let it be termed an *autopolar* of the fundamental cubic; and there will arise the problem, to determine conic autopolars, cubic autopolars, etc. A conic autopolar will have as defining equation

$$(aa'u)^2 a_a a'_a \equiv \mu \cdot u_a^2, \tag{12}$$

or

$$2\begin{vmatrix} a_1 & ma_3 & ma_2 & u_1 \\ ma_3 & a_2 & ma_1 & u_2 \\ ma_2 & ma_1 & a_3 & u_3 \\ u_1 & u_2 & u_3 & 0 \end{vmatrix} \equiv \mu \cdot u_a^2 .$$

The six conditions of this identity fall into three pairs as before and determine two discrete ∞^2 webs of conics. From one pair of conditions:

$$\left\{\begin{array}{ll} 2m^2 a_{11} - 2a_{23} & = \mu a_{11}, \\ 4ma_{11} - 4m^2 a_{23} & = 2\mu a_{23}, \end{array}\right\} \tag{13}$$

eliminate μ, set $a_{23}/a_{11} = \alpha$, and call the roots of the resulting quadric α and β, *i. e.*,

$$\left\{\begin{array}{l} \alpha^2 - 2m^2\alpha + m = 0, \\ \beta^2 - 2m^2\beta + m = 0. \end{array}\right\} \tag{14}$$

The two systems of conics and the cubics whose polar systems they are respectively have the equations

$$(15)\qquad \left.\begin{aligned}(v_1u_1^2+v_2u_2^2+v_3u_3^2)+2\alpha\cdot(v_1u_2u_3+v_2u_3u_1+v_3u_1u_2)=0,\\(v_1u_1^2+v_2u_2^2+v_3u_3^2)+2\beta\cdot(v_1u_2u_3+v_2u_3u_1+v_3u_1u_2)=0,\end{aligned}\right\}$$

the v's being arbitrary quantities, and

$$(16)\qquad \left.\begin{aligned}u_\alpha^3=u_1^3+u_2^3+u_3^3+6\alpha u_1u_2u_3=0,\\u_\beta^3=u_1^3+u_2^3+u_3^3+6\beta u_1u_2u_3=0.\end{aligned}\right\}$$

From (14) we find the relation between α and β to be

$$(17)\qquad 2\alpha^2\beta^2-(\alpha+\beta)=0.$$

If α is considered constant, β and γ may denote roots of this equation, and the relation between them is found to be of the same form:

$$(17a)\qquad 2\beta^2\gamma^2-(\beta+\gamma)=0.$$

These three again constitute a closed set. The parameters can be separated, giving the equal rational functions:

$$-\frac{1+2\alpha^3}{6\alpha^2}=-\frac{1+2\beta^3}{6\beta^2}=-\frac{1+2\gamma^3}{6\gamma^2}=C',$$

and this is the parameter of the Hessian* of each of the cubics (16). Notice also that the unsymmetric relation (14) becomes the symmetric (17a) if we take $\gamma=-1/2m$.

The two systems of conic autopolars of a plane cubic are the conic polars of two curves of third class which have a common Hessian; and this is also the Hessian of the curve of third class which is "conjugate" or doubly apolar to the fundamental curve of third order.

The term *conjugate*, introduced by HESSE (l. c.), has been applied more recently to curves which are simply apolar. The curves here considered have all conic polars of one apolar to all conic polars of the other; hence the term *doubly apolar* seems to me preferable. It is easily verified that any polar of the one,

$$u_1^2-\frac{2}{2m}u_2u_3=0,$$

is apolar to three independent conic polars of the other,

$$\begin{aligned}x_1^2+2m\,x_2x_3=0,\\x_2^2+2m\,x_3x_1=0,\\x_3^2+2m\,x_1x_2=0.\end{aligned}$$

* See SALMON, *Higher Plane Curves* (3d ed.), p. 190.

The two webs of conic autopolars are connected in this same manner with the two nets of autopoloconics: that is to say, each web is apolar to one of the nets, and is autopolar to the cubic of the other net. This is already evident geometrically from the fact that the Cayleyan of any cubic is the Hessian of its doubly apolar cubic.* Otherwise the mutual apolarity may be readily verified in the following manner from the characteristic equations of conics in the net and web respectively. Let A_x^2 and u_a^2 satisfy relations (2) and (12):

$$(aa'A)^2 a_x a'_x \equiv \lambda_A \cdot A_x^2,$$

$$(aa'u)^2 a_a a'_a \equiv \mu_a \cdot u_a^2,$$

and let μ_a denote the value of μ *different from* λ_A. Then operating on both members of these identities with

$$\sum_{i,k}\left(a_{ik}\frac{\partial^2}{\partial x_i \partial x_k}\right), \quad \sum_{i,k}\left(A_{ik}\frac{\partial^2}{\partial u_i \partial u_k}\right),$$

we shall find

$$(aa'A)^2 a_a a'_a \equiv \lambda_A \cdot A_a^2,$$

$$(aa'A)^2 a_a a'_a \equiv \mu_a \cdot A_a^2;$$

or since $\lambda_A + \mu_a$ (indeed $\mu_a = -\lambda_A$), we have by subtraction:

$$A_a^2 = 0. \quad \text{Similarly } B_\beta^2 = 0. \tag{18}$$

Remembering that these conics are polars of cubics, we may state the relations of the cubics as follows:

The two point cubics $A_x^3 = 0$ *and* $B_x^3 = 0$ *satisfying the conditions for autopoloconics*

$$(aa'A)^2 a_x a'_x A_y \equiv \quad \lambda \cdot A_x^2 A_y,$$

$$(aa'B)^2 a_x a'_x B_y \equiv -\lambda \cdot B_x^2 B_y,$$

are respectively doubly apolar to the two line cubics $u_a^3 = 0$ *and* $u_\beta^3 = 0$ *which satisfy the conditions for conic autopolars*

$$(aa'u)^2 a_a a'_a v_a \equiv -\lambda \cdot u_a^2 v_a,$$

$$(aa'u)^2 a_\beta a'_\beta v_\beta \equiv \quad \lambda \cdot u_\beta^2 v_\beta.$$

In other words, the two equations subsist identically:

$$A_a^2 A_x u_a \equiv 0,$$

$$B_\beta^2 B_x u_\beta \equiv 0. \tag{19}$$

Moreover the conic polars of the curves A, B, α, and β are autopoloconics respectively to the curves B, A, β, and α, and conic autopolars respectively to the curves

* CLEBSCH-LINDEMANN, *Vorlesungen über Geometrie*, vol. I., p. 521.

β, a, B and A. The conic polars of the fundamental cubic a are autopoloconics of A and B, and conic autopolars to the curves a and β. That is to say, the following characteristic relations are fulfilled:

$$
\begin{aligned}
&(AA'B)^2 A_x A'_x B_y \equiv \pi \cdot B_x^2 B_y, \\
&(BB'A)^2 B_x B'_x A_y \equiv \rho \cdot A_x^2 A_y, \\
&(a\alpha'\beta)^2 u_a u_{\alpha'} v_\beta \equiv \sigma \cdot u_\beta^2 v_\beta, \\
&(\beta\beta'a)^2 u_\beta u_{\beta'} v_a \equiv \tau \cdot u_a^2 v_a; \\
&(AA'u)^2 A_\beta A'_\beta v_\beta \equiv -\pi \cdot u_\beta^2 v_\beta, \\
&(BB'u)^2 B_a B'_a v_a \equiv -\rho \cdot u_a^2 v_a, \\
&(a\alpha'x)^2 B_a B_{\alpha'} B_y \equiv -\sigma \cdot B_x^2 B_y, \\
&(\beta\beta'x)^2 A_\beta A_{\beta'} A_y \equiv -\tau \cdot A_x^2 A_y.
\end{aligned} \tag{20}
$$

§3. *Defining equations for two order-cubics and two class-cubics related to the Hessian.*

Denote by $f = a_x^3 = 0$ the fundamental cubic, and by $\pi = u_\pi^3 = 0$ its doubly apolar curve of third class; we may then speak of the curve f and the curve π. It is known that an interchange of the curves f and π interchanges also the Hessian and the Cayleyan, the point-coördinates and line-coördinates, etc. Accordingly by forming the covariant Θ for π instead of for f, we shall be able to define algebraically as irrational covariants those systems of conics and their cubics which are related to the Hessian just as those already discussed were related to the Cayleyan. For this purpose, take from CLEBSCH and GORDAN'S admirable paper in Mathematische Annalen, vol. 6, two formulæ,

$$\pi = u_\pi^3 = S\mathrm{T} - T\Sigma, \tag{21}$$

$$\Theta^{(\pi)} = (\pi\pi' x)^2 u_\pi u_{\pi'} = R(\mathrm{K} - \tfrac{1}{6} S\Theta), \tag{22}$$

(particularized from (91), p. 494), where K denotes the covariant Θ formed for the Hessian Δ_x^3 instead of the curve f or a_x^3, viz.,

$$\mathrm{K} = (\Delta\Delta' u)^2 \Delta_x \Delta'_x,$$

and R denotes the discriminant of the cubic,

$$R = T^2 - \tfrac{1}{6} S^3.$$

By omitting the factor R, as our cubic is assumed to be non-singular, these changes give us the theorem:

The two point-cubics having the same Hessian with a given fundamental cubic $a_x^3 = 0$ *being denoted by equations* $F_x^3 = 0$ *and* $G_x^3 = 0$; *and the two line-cubics respectively doubly apolar to these, having therefore this Hessian as their common Cayleyan curve, being denoted by equations* $u_\phi^3 = 0$ *and* $u_\psi^3 = 0$, *these two pairs of curves are completely defined by the covariant equations:*

$$(23)\qquad \begin{aligned} (\Delta\Delta' F)^2 F_y \Delta_x \Delta'_x - \tfrac{1}{6} S(aa' F)^2 F_y a_x a'_x &\equiv \quad \mu \cdot F_x^2 F_y, \\ (\Delta\Delta' G)^2 G_y \Delta_x \Delta'_x - \tfrac{1}{6} S(aa' G)^2 G_y a_x a'_x &\equiv -\mu \cdot G_x^2 G_y, \end{aligned}$$

$$(24)\qquad \begin{aligned} (\Delta\Delta' u)^2 v_\phi \Delta_\phi \Delta'_\phi \ - \tfrac{1}{6} S(aa' u)^2 v_\phi a_\phi a'_\phi \ &\equiv -\mu \cdot u_\phi^2 v_\phi, \\ (\Delta\Delta' u)^2 v_\psi \Delta_\psi \Delta'_\psi \ - \tfrac{1}{6} S(aa' u)^2 v_\psi a_\psi a'_\psi \ &\equiv \quad \mu \cdot u_\psi^2 v_\psi, \end{aligned}$$

where μ *is a constant determinable from the equations.*

To these might be appended also the sets analogous to those in (20), but the geometric relations are already sufficiently evident.

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FORMENTHEORETISCHE ENTWICKELUNG DER IN HERRN WHITE'S ABHANDLUNG ÜBER CURVEN DRITTER ORDNUNG ENTHALTENEN SÄTZE*

VON

PAUL GORDAN

§1. *Covarianten und Invarianten einer ternären kubischen Form.*

Die einfachsten invarianten Bildungen einer ternären kubischen Form

$$f = a_x^3$$

sind diese 10 Formen:

$$\theta = \theta_x^2 u_\vartheta^2 = (abu)^2 a_x b_x\,; \quad \Delta = \Delta_x^3 = a_\vartheta^2 a_x \theta_x^2\,; \quad s = u_s^3 = (a\theta u)^2 a_\vartheta u_\vartheta\,;$$

$$\mathrm{H} = \mathrm{H}_x^2 u_\eta^2 = (a\Delta u)^2 a_x \Delta_x\,; \quad \mathrm{K} = \mathrm{K}_x^2 u_\varkappa^2 = (\Delta\Delta_1 u)^2 \Delta_x \Delta_{1,x}\,; \quad t = u_t^3 = (a\mathrm{H}u)^2 a_\eta u_\eta\,;$$

$$S = a_s^3\,; \quad T = \Delta_s^3 = a_t^3\,; \quad \pi = u_\pi^3 = Ts - St\,; \quad R = T^2 - \tfrac{1}{6} S^3.$$

Von den zahlreichen zwischen ihnen bestehenden Relationen erwähne ich zunächst diese 4:

$$(1) \qquad a_s^2 a_x u_s = \tfrac{1}{3} S \cdot u_x\,; \quad a_t^2 u_t a_x = \tfrac{1}{3} T \cdot u_x\,; \quad \Delta_s^2 u_s \Delta_x = \tfrac{1}{3} T \cdot u_x\,;$$
$$\Delta_t^2 u_t \Delta_x = \tfrac{1}{18} S^2 \cdot u_x\,,$$

und die 4 aus ihnen folgenden:

$$(2) \qquad (\vartheta s x)^2 \theta_s \theta_x = \tfrac{2}{3} S \cdot f\,; \quad (\varkappa s x)^2 \mathrm{K}_s \mathrm{K}_x = \tfrac{2}{3} T \cdot s\,;$$
$$a_\pi^2 u_\pi a_x \equiv 0\,; \quad \Delta_\pi^2 \Delta_x u_\pi = R \cdot u_x\,.$$

§2. *Die Combinanten des Polarnetzes von f.*

Die Polaren $a_x^2 a_y$ von f bilden ein Kegelschnittnetz:

$$P(f).$$

Die Hauptcombinanten sind die Elementar-Covarianten von

$$(abc)\, a_x^2 b_y^2 c_z^2\,,$$

* Presented to the Society Oct. 28, 1899. Received for publication Oct. 15, 1899.

also nach der Formel

$$3(abc)a_x^2b_y^2c_z^2 = 2(xyz)\Delta_x\Delta_y\Delta_z - (sxy)(sxz)(syz)$$

die Hessische Form Δ und die Cayley'sche Form s; die anderen Combinanten von $P(f)$ sind die Formen des simultanen Systems von Δ und s. Die erste der Elementar-Covarianten von $(abc)a_x^2b_y^2c_z^2$ ist

$$(abc)(bcu)a_y^2b_xc_x = a_\vartheta u_\vartheta a_y^2\theta_x^2;$$

sie hat den Werth

$$a_\vartheta u_\vartheta a_y^2\theta_x^2 = \tfrac{2}{3}u_y\Delta_x^2\Delta_y - \tfrac{1}{3}u_x\Delta_x\Delta_y^2 + \tfrac{1}{3}u_s(sxy)^2. \tag{3}$$

§ 3. *Apolare Netze.*

In Folge der Formel

$$a_x^2a_xu_\pi = 0$$

sind die Netze $P(f)$ und $P(\pi)$ apolar. Die Coefficienten-Matrices beider Netze sind korrespondirend und es besteht die Relation:

$$(\pi\pi_1\pi_2)(\pi xy)(\pi xz)(\pi_1xz)(\pi_1yz)(\pi_2yz)(\pi_2xy) = \rho(xyz)(abc)a_x^2b_y^2c_z^2;$$

sie sagt aus, f und π haben dieselben Haupt-Combinanten. Die Hessische Curve von f ist die Cayley'sche von π und die Hessische Curve von π die Cayley'sche von f. Die Untersuchung * des Herrn WHITE geht dahin Curven zu finden, welche entweder die Cayley'sche oder die Hessische mit f gemein haben.

§ 4. *Relationen zwischen den invarianten Bildungen von f.*

Die Relationen zwischen den invarianten Bildungen von f folgen im Wesentlichen aus Formel (3); ich erwähne hier nur diese 10 einfachsten:

$$\begin{gathered}
\mathrm{H}_\vartheta^2\theta_x^2u_\eta^2 = -\tfrac{1}{3}\,\mathrm{K} + \tfrac{1}{3}\,\Delta_s\Delta_x^2u_s^2 + \tfrac{1}{9}T\cdot u_x^2;\\
a_su_s^2a_x^2 = \mathrm{H} + \tfrac{1}{6}\,S\cdot u_x^2;\\
\theta_s^2u_su_\vartheta^2 = t;\\
\Delta_s\Delta_x^2u_s^2 = -\tfrac{1}{2}\,\mathrm{K} + \tfrac{1}{4}\,S\cdot\theta + \tfrac{1}{6}\,T\cdot u_x^2;\\
\mathrm{K}_x^2u_x^2u_s = \tfrac{1}{2}\,St - \tfrac{1}{3}\,Ts;\\
\theta_i^2u_iu_\vartheta^2 = \tfrac{1}{6}\,Ss;\\
\theta_s = (ss_1x)^2\,u_su_{s_1} = \mathrm{K} + \tfrac{1}{2}\,S\theta;\\
\Delta_s = (ss_1s_2)^2\,u_su_{s_1}u_{s_2} = St - \tfrac{1}{3}\,Ts = \tfrac{2}{3}\,Ts - \pi;\\
s_s = (ss_1s_2)\,(ss_1x)\,(ss_2x)\,(s_1s_2x) = \tfrac{1}{3}\,S^2f + \tfrac{2}{3}\,T\Delta;\\
S_s = (ss_1s_2)\,(ss_1s_3)\,(ss_2s_3)\,(s_1s_2s_3) = \tfrac{1}{2}\,S^3 + \tfrac{3}{2}\,T^2.
\end{gathered} \tag{4}$$

* S. pag. 1 dieses Heftes.

§ 5. *Der Aronhold'sche Prozess.*

Die invarianten Bildungen des Aggregates

$$xf + \lambda\Delta$$

werden mittelst des Aronhold'schen Prozesses

$$\partial f = \Delta$$

berechnet; wendet man ihn auf die einfachsten Formen an, so erhält man:

$$\partial\Delta = \tfrac{1}{2} S\cdot f; \quad \partial s = 3t; \quad \partial S = 4T; \quad \partial t = \tfrac{5}{6} S\cdot s;$$
$$\partial T = S^2; \quad \partial\theta = 2\mathrm{H}; \quad \partial\mathrm{H} = \mathrm{K} + \tfrac{1}{2} S\theta; \quad \partial\mathrm{K} = S\mathrm{H}; \quad \partial R = 0.$$

§ 6. *Einige invariante Bildungen von*

$$xf + \lambda\Delta, \quad xs + \lambda t, \quad x\pi + \lambda s.$$

Die Werthe der Hessischen Form und der Cayley'schen Form von

$$xf + \lambda\Delta$$

sind bekanntlich:

$$\Delta_{xf+\lambda\Delta} = \frac{1}{2S}\lambda(Sx + (T + \sqrt{R})\lambda)\,(Sx + (T - \sqrt{R})\lambda)f + (x^3 - \tfrac{1}{2}Sx\lambda^2 - \tfrac{1}{3}T\lambda^3)\Delta; \tag{5}$$

$$s_{xf+\lambda\Delta} = 3\lambda(x - \sqrt{\tfrac{1}{6}S}\cdot\lambda)\,(x + \sqrt{\tfrac{1}{6}S}\cdot\lambda)t + (x^3 + \tfrac{1}{2}Sx\lambda^2 + \tfrac{2}{3}T\cdot\lambda^3)s. \tag{6}$$

Analoge Formeln erhält man durch Anwendung des Aronhold'schen Prozesses auf die Formeln (4):

$$\theta_{xs+\lambda t} = (x^2 - \tfrac{1}{6}S\lambda^2)\mathrm{K} + (\tfrac{2}{3}Sx\lambda + \tfrac{2}{3}T\lambda^2)\mathrm{H} + (\tfrac{1}{2}Sx^2 + \tfrac{2}{3}Tx\lambda + \tfrac{1}{36}S^2\lambda^2)\theta;$$
$$\theta_{x\pi+\lambda s} = (Rx^2 + 2Tx\lambda + \lambda^2)\mathrm{K} - \tfrac{2}{3}S^2x\lambda\cdot\mathrm{H} + (-\tfrac{1}{6}SRx^2 + \tfrac{1}{3}STx\lambda + \tfrac{1}{2}S\lambda^2)\theta;$$

$$\Delta_{xs+\lambda t} = (Sx + T\lambda)(x + \sqrt{\tfrac{1}{6}S}\cdot\lambda)(x - \sqrt{\tfrac{1}{6}S}\cdot\lambda)\,t + (-\tfrac{1}{3}Tx^3 + \tfrac{1}{6}S^2x^2\lambda + \tfrac{1}{2}STx\lambda^2 + (\tfrac{2}{9}T^2 + \tfrac{1}{108}S^3)\lambda^3)s; \tag{7}$$

$$\Delta_{x\pi+\lambda s} = (-\tfrac{1}{3}R^2x^3 + Rx\lambda^2 + \tfrac{2}{3}T\lambda^3)s - (Rx^2\lambda + 2Tx\lambda^2 + \lambda^3)\pi; \tag{8}$$

$$s_{x\pi+\lambda s} = \tfrac{1}{3}S^2\lambda(\lambda - \sqrt{R}\cdot x)(\lambda + \sqrt{R}\cdot x)f + (\tfrac{2}{3}R^2x^3 + 2RTx^2\lambda + 2Rx\lambda^2 + \tfrac{2}{3}T\lambda^3)\Delta; \tag{9}$$

$$\tfrac{8}{2}S_{x\pi+\lambda s} = R^3x^4 + 4R^2Tx^3\lambda + 6R^2x^2\lambda^2 + 4RTx\lambda^3 + (\tfrac{1}{2}S^3 + T^2)\lambda^4. \tag{10}$$

§ 7. *Curven, welche dieselbe Hessische und welche dieselbe Cayley'sche Curve haben.*

Die Formeln (9) und (8)

$$s_\pi = \tfrac{2}{3}R^2\Delta, \quad \Delta_\pi = -\tfrac{1}{3}R^2s$$

bestätigen die im § 3 gemachte Aussage, dass die Cayley'sche Curve von π die Hessische von f und dass die Hessische Curve von π die Cayley'sche von f ist.

Diejenigen Curven, welche mit f oder π die Cayley'sche oder die Hessische Curve gemein haben sind covariant zu f und gehören daher den Büscheln an

$$xf + \lambda\Delta, \quad xs + \lambda t \text{ resp. } x\pi + \lambda s.$$

Ihre Parameter erhält man durch das Verschwinden der Coefficienten von f, t, π in den Formeln (5) bis (9). Da dieselben kubisch sind, so giebt es von jeder Art 3 Curven, ein Curventripel. Zur Cayley'schen Curve s gehört das Tripel:

$$(11) \qquad f; \quad A = \Delta + \sqrt{\tfrac{1}{6}S}\cdot f; \quad B = \Delta - \sqrt{\tfrac{1}{6}S}\cdot f;$$

zur Hessischen Curve Δ das Tripel:

$$(12) \quad f; \quad F = S\Delta + (\sqrt{R} - T)f; \quad G = S\Delta - (\sqrt{R} + T)f;$$

zur Cayley'schen Curve Δ das Tripel:

$$(13) \qquad \pi; \quad \Phi = \pi + \sqrt{R}\cdot s; \quad \Psi = \pi - \sqrt{R}\cdot s;$$

und zur Hessischen Curve s das Tripel:

$$(14) \qquad \pi; \quad A = t + \sqrt{\tfrac{1}{6}S}\cdot s; \quad B = t - \sqrt{\tfrac{1}{6}S}\cdot s.$$

§ 8. *Die Formen* Δ *und* s *der Tripel.*

Die Cayley'schen Formen des Tripels:

$$f, \quad A, \quad B$$

haben die Werthe:

$$s_f = s; \quad s_A = \tfrac{2}{3}(T + \sqrt{\tfrac{1}{6}S^3})s; \quad s_B = \tfrac{2}{3}(T - \sqrt{\tfrac{1}{6}S^3})s;$$

die Cayley'schen des Tripels:

$$\pi, \quad \Phi, \quad \Psi$$

die Werthe:

$$s_\pi = \tfrac{2}{3}R^2\Delta; \quad s_\Phi = \tfrac{8}{3}\sqrt{R^3}(T - \sqrt{R})\Delta; \quad s_\Psi = -\tfrac{8}{3}\sqrt{R^3}(T + \sqrt{R})\Delta;$$

die Hessischen des Tripels:

$$f, \quad F, \quad G$$

die Werthe:

$$\Delta_f = \Delta; \quad \Delta_F = -4R(T + \sqrt{R})\Delta; \quad \Delta_G = -4R(T - \sqrt{R})\Delta;$$

und die Hessischen des Tripels:

$$\pi, \quad A, \quad B$$

die Werthe:

$$\Delta_\pi = -\tfrac{1}{3}R^2 s; \quad \Delta_A = \tfrac{2}{9}(T + \sqrt{\tfrac{1}{6}S^3})^2 s; \quad \Delta_B = \tfrac{2}{9}(T - \sqrt{\tfrac{1}{6}S^3})^2 s.$$

§ 9. *Die Formen* $t,\ T,\ A,\ B,\ \mathrm{A},\ \mathrm{B}$ *von* π.

Die Formen

$$t,\ T,\ A,\ B,\ \mathrm{A},\ \mathrm{B}$$

gebildet für die Form π haben die Werthe:

$$t_\pi = -\tfrac{2}{9} R^3 (T\Delta - \tfrac{1}{6} S^2 f); \qquad T_\pi = -\tfrac{2}{9} R^4 T;$$

$$A_\pi = \tfrac{1}{3}\sqrt{R^3}\cdot\Psi; \qquad B_\pi = -\tfrac{1}{3}\sqrt{R^3}\,\Phi;$$

$$\mathrm{A}_\pi - = \frac{2}{9S} R^3 (T + \sqrt{R})\, F; \qquad \mathrm{B}_\pi = -\frac{2}{9S} R^3 (T - \sqrt{R})\, G.$$

§ 10. *Apolarität der Netze* $P(A)$ *und* $P(\mathrm{B})$.

In Folge der Formel

$$A_\mathrm{B}^2 A_x u_\mathrm{B} = 0$$

sind die Netze $P(A)$ und $P(\mathrm{B})$ apolar. Die einfachsten invarianten Bildungen der Form A haben die Werthe;

$$s_A = \tfrac{2}{3}(T + \sqrt{\tfrac{1}{6}S^3})s; \qquad \theta_A = \tfrac{1}{6} S\theta + 2\sqrt{\tfrac{1}{6}S}\cdot \mathrm{H} + \mathrm{K};$$

$$t_A = \tfrac{4}{9} S (T + \sqrt{\tfrac{1}{6}S^3})\, t + \tfrac{2}{9} Rs.$$

$$S_A = \tfrac{2}{3}(T + \sqrt{\tfrac{1}{6}S^3})^2; \quad T_A = \tfrac{4}{9}\sqrt{\tfrac{1}{6}S^3}\cdot(T + \sqrt{\tfrac{1}{6}S^3})^2 - \tfrac{2}{9} R (T + \sqrt{\tfrac{1}{6}S^3});$$

$$\pi_A = -\tfrac{8}{27} S (T + \sqrt{\tfrac{1}{6}S^3})^3 \mathrm{B}.$$

Die letzte Formel bestätigt die Apolarität.

§ 11. *Die charakteristischen Relationen für das Tripel* $f,\ A,\ B$.

Die zwischen $f,\ A,\ B,$ bestehenden Relationen:

$$(abA)^2 a_x b_x A_y = \sqrt{\tfrac{1}{6}S}\cdot A_x^2 A_y;$$

$$(AA_1a)^2 A_x A_{1,x} a_y = \tfrac{1}{3}(T + \sqrt{\tfrac{1}{6}S^3})\, a_x^2 a_y;$$

$$(AA_1B)^2 A_x A_{1,x} B_y = -\tfrac{1}{3}(T + \sqrt{\tfrac{1}{6}S^3}) B_x^2 B_y$$

charakterisiren das Tripel $f,\ A,\ B$. Die Formel:

$$(abL)^2 a_x b_x = \sqrt{\tfrac{1}{6}S}\cdot L_x^2 \tag{15}$$

liefert 6 Relationen zwischen den Coefficienten des Kegelschnittes L; 3 derselben folgen nach der Formel:

$$(abL)^2 a_A b_A u_A = \sqrt{\tfrac{1}{6}S}\cdot L_A^2 u_A$$

aus den andern. Das durch Formel (15) charakterisirte Kegelschnittnetz ist $P(A)$.

SUR LA DÉFINITION GÉNÉRALE DES FONCTIONS ANALYTIQUES, D'APRÈS CAUCHY*

PAR

E. GOURSAT

J'ai reconnu depuis longtemps que la démonstration du théorème de Cauchy, que j'ai donnée en 1883†, ne supposait pas la continuité de la dérivée. Pour répondre au désir qui m'a été exprimé par M. le Professeur Wm. F. Osgood, je vais indiquer ici rapidement comment on peut faire cette extension.

Soit z une variable complexe et $u = f(z)$ une autre quantité complexe qui varie avec z. La fonction $f(z)$ est dite *continue* pour la valeur z_0 de la variable si la différence $f(z_0 + h) - f(z_0)$ tend vers zéro en même temps que le module de h, ou, d'une façon plus précise, si à tout nombre positif ε, pris arbitrairement, on peut faire correspondre un autre nombre positif η, de telle façon que l'inégalité

$$|h| < \eta$$

entraîne la suivante:

$$|f(z_0 + h) - f(z_0)| < \varepsilon.$$

Une fonction continue $f(z)$ admet, pour $z = z_0$, une dérivée $f'(z_0)$ si le rapport

$$\frac{f(z_0 + h) - f(z_0)}{h}$$

tend vers $f'(z_0)$ lorsque le module de h tend vers zéro ; cela revient encore à dire qu' à tout nombre positif ε, pris arbitrairement, on peut faire correspondre un autre nombre positif η tel que l'inégalité

$$|h| < \eta$$

entraîne la suivante:

$$|f(z_0 + h) - f(z_0) - h f'(z_0)| \leqq |h|\ \varepsilon.$$

Ces définitions étant rappelées, je definirai encore une expression que j'emploierai, pour abréger la démonstration. Soit A une portion du plan limitée par un contour fermé C, et $f(z)$ une fonction continue et admettant une dérivée

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† Acta Mathematica, tome 4, p. 197–200; Comptes Rendus de l' Académie des Sciences de Stockholm, 1884.

pour chaque point de l'aire A et du contour C; soit d'autre part ε un nombre positif. Je dirai que le contour C satisfait à la condition (α) relativement au nombre ε s'il est possible de trouver à l'intérieur ou sur le contour C un point fixe z' tel que l'on ait

$$|f(z) - f(z') - (z - z')f'(z')| \leqq |z - z'|\,\varepsilon,$$

lorsque z décrit le contour C.

Cela posé, toute de la démonstration repose sur le lemme préliminaire suivant:

LEMME.—*Soient $f(z)$ une fonction continue et admettant une dérivée pour tous les points d'une aire A limitée par un contour fermé C et de ce contour lui-même, et ε un nombre positif arbitraire. On peut toujours décomposer l'aire A en portions assez petites pour que le contour de chacune de ces portions satisfasse à la condition (α) relativement au nombre positif ε.*

On établit ce lemme par le procédé bien connu de subdivisions successives. Pour fixer les idées, je supposerai que l'on décompose l'aire A en parties plus petites par deux séries de droites parallèles, à l'axe réel d'une part, à la perpendiculaire d'autre part, la distance de deux parallèles voisines étant constante. Si l'aire A ne satisfait pas à l'énoncé du lemme, il y aura au moins une des aires partielles A_1 qui n'y satisfera pas non plus. En subdivisant cette aire A_1 par le même procédé, on en déduira une aire plus petite A_2, et ainsi de suite. Le procédé pouvant se continuer indéfiniment, on a une suite d'aires

$$A_1, A_2, \cdots A_n, \cdots$$

dont chacune est comprise dans la précédente, et dont les deux dimensions tendent vers zéro, lorsque n augmente indéfiniment. Il y a donc un point limite z_0, intérieur à A ou situé sur le contour C. Puisque, par hypothèse, la fonction $f(z)$ admet une dérivée $f'(z_0)$ pour $z = z_0$, on peut trouver un nombre η tel que l'on ait

$$|f(z) - f(z_0) - (z - z_0)f'(z_0)| \leqq |z - z_0|\,\varepsilon,$$

pourvu que $|z - z_0|$ soit $< \eta$. Soit c le cercle de rayon η décrit du point z_0 comme centre. A partir d'une valeur de n assez grande, l'aire A_n sera intérieure au cercle c, et on aura pour tous les points du contour de l'aire A_n

$$|f(z) - f(z_0) - (z - z_0)f'(z_0)| \leqq |z - z_0|\,\varepsilon.$$

D'ailleurs il est clair que le point z_0 est à l'intérieur de A_n ou sur le contour; cette aire devrait donc satisfaire à la condition (α) relativement à ε. Nous sommes par conséquent conduits à une contradiction en admettant que le lemme n'est pas exact.

Il suffit maintenant de supposer, dans ma démonstration du théorème de Cauchy, que l'on choisit les points z_i de façon que les modules de toutes les quantités ε_i soient moindres qu'un nombre donné ε, sans modifier en rien le reste du raisonnement.

Du théorème sur l'intégrale $\int f(z)dz$, on déduira ensuite la formule fondamentale

$$2\pi i \cdot f(x) = \int_{(C)} \frac{f(z)dz}{z-x},$$

avec toutes ses conséquences. On voit par là qu'en se plaçant au point de vue de CAUCHY il suffit, pour édifier la thèorie des fonctions analytiques, de supposer la continuité de $f(z)$ et l'existence de la dérivée.

ON A CLASS OF PARTICULAR SOLUTIONS OF THE PROBLEM OF FOUR BODIES*

BY

FOREST RAY MOULTON

Introduction. In 1772 LAGRANGE published a celebrated memoir on the Problem of Three Bodies, which contained all the solutions in which the ratios of the mutual distances of the bodies are constants. He found two distinct configurations. In the one, the three bodies always lie in a straight line; in the other, they are always at the vertices of an equilateral triangle. Their distribution upon the line depends upon their masses, being explicitly defined by the real positive root of a certain quintic equation. The equilateral triangular configuration is possible for all distributions of the masses. In both cases the three bodies move in the same plane, in conic sections with respect to each other or with respect to their common center of gravity, and in such a manner that the law of areas is true for each body considered separately.

No other periodic solutions of the motion of three or more bodies were discovered for more than a century, although many splendid papers appeared on the Problem of Three Bodies. A new impetus was given to the subject by the celebrated memoir of Dr. HILL on the Lunar Theory,† in which he discussed a new species of periodic solutions. He made the restrictions that one body should be infinitesimal, and that the finite bodies should describe circles around their center of gravity. For the purposes of the applications to the Lunar Theory he neglected the ratio of the distances of the moon and the sun, thus introducing errors in the numerical results which in certain cases became important.

Professor DARWIN has more recently taken up the subject‡ by a method not essentially different from that employed by HILL, and discussed a great many orbits in detail, not neglecting the solar parallax. He has considered only cases in which the infinitesimal body moves in the plane of the finite bodies with direct motion.

To these must be added the masterful researches § of POINCARÉ, who has

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† American Journal of Mathematics, vol. 1.

‡ Acta Mathematica, vol. 21.

§ *Les Méthodes Nouvelles de la Mécanique Céleste*, I and III.

proved the existence of several classes of infinite numbers of periodic solutions of the Problems of Three Bodies. His results are not given explicitly, and not all of the circumstances of the motion are known as they are in the previous cases.

This paper treats of the Problem of Four Bodies, three of which are finite, moving in *circles* according to one or the other of the solutions of LAGRANGE, while the fourth is infinitesimal. The motion of the systems is referred to axes rotating with uniform angular velocity. An integral, similar to that of JACOBI in the Problem of Three Bodies, is found, which leads to the definition of the surfaces of zero relative velocity. For certain initial conditions these surfaces have double points, all of which are found.

It is shown that if the infinitesimal body be started at any double point of the surfaces, with zero relative velocity, it will always remain in the same position with respect to the finite bodies. Therefore they are particular solutions of the motion of the system, and it is true that they are the only ones in which the mutual distances of the bodies are constants. There are eighteen solutions of arbitrary periods of revolution in which the finite bodies lie in a line, and ten in which they are at the vertices of an equilateral triangle.

§ 1. *The differential equations of motion.*

Suppose the masses of the finite bodies are m_1, m_2, and m_3 respectively. Suppose the origin is taken at the center of gravity of the system. Let the coördinates of m_i be ξ_i, η_i, ζ_i, $i = 1, 2, 3$, and of the infinitesimal body, ξ, η, and ζ. Suppose the units are chosen so that the Gaussian constant is equal to unity. Let

$$\rho_i = \sqrt{(\xi - \xi_i)^2 + (\eta - \eta_i)^2 + (\zeta - \zeta_i)^2} \qquad (i = 1, 2, 3)$$

and

$$U = \frac{m_1}{\rho_1} + \frac{m_2}{\rho_2} + \frac{m_3}{\rho_3}.$$

Then the differential equations of motion of the infinitesimal body are

$$(1) \qquad \begin{cases} \dfrac{d^2\xi}{dt^2} = \dfrac{\partial U}{\partial \xi}, \\ \dfrac{d^2\eta}{dt^2} = \dfrac{\partial U}{\partial \eta}, \\ \dfrac{d^2\zeta}{dt^2} = \dfrac{\partial U}{\partial \zeta}. \end{cases}$$

The motion of the system is referred to xy-axes rotating in the $\xi\eta$-plane by the substitution

$$(2)\qquad \begin{cases} \xi_i = x_i \cos\theta - y_i \sin\theta, \\ \eta_i = x_i \sin\theta + y_i \cos\theta, \\ \zeta_i = z_i, \end{cases}$$

where θ is a function of the time.

Equation (1) is transformed by this substitution into

$$(3)\qquad \begin{cases} \dfrac{d^2x}{dt^2} - 2\dfrac{d\theta}{dt}\dfrac{dy}{dt} - \left(\dfrac{d\theta}{dt}\right)^2 x - \dfrac{d^2\theta}{dt^2}\, y = \dfrac{\partial U}{\partial x}, \\ \dfrac{d^2y}{dt^2} + 2\dfrac{d\theta}{dt}\dfrac{dx}{dt} - \left(\dfrac{d\theta}{dt}\right)^2 y + \dfrac{d^2\theta}{dt^2}\, x = \dfrac{\partial U}{\partial y}, \\ \dfrac{d^2z}{dt^2} = \dfrac{\partial U}{\partial z}. \end{cases}$$

If the axes rotate with the uniform angular velocity, n, the system (3) becomes

$$(4)\qquad \begin{cases} \dfrac{d^2x}{dt^2} - 2n\dfrac{dy}{dt} = n^2x + \dfrac{\partial U}{\partial x} = n^2x - \dfrac{m_1(x-x_1)}{\rho_1^3} \\ \qquad\qquad - \dfrac{m_2(x-x_2)}{\rho_2^3} - \dfrac{m_3(x-x_3)}{\rho_3^3}, \\ \dfrac{d^2y}{dt^2} + 2n\dfrac{dx}{dt} = n^2y + \dfrac{\partial U}{\partial y} = n^2y - \dfrac{m_1(y-y_1)}{\rho_1^3} \\ \qquad\qquad - \dfrac{m_2(y-y_2)}{\rho_2^3} - \dfrac{m_3(y-y_3)}{\rho_3^3}, \\ \dfrac{d^2z}{dt^2} = \dfrac{\partial U}{\partial z} = -\dfrac{m_1(z-z_1)}{\rho_1^3} - \dfrac{m_2(z-z_2)}{\rho_2^3} - \dfrac{m_3(z-z_3)}{\rho_3^3}. \end{cases}$$

The substitution

$$(5)\qquad \begin{cases} \dfrac{dx}{dt} - ny = x', \\ \dfrac{dy}{dt} + nx = y', \\ \dfrac{dz}{dt} = z', \\ H = \frac{1}{2}(x' + ny)^2 + \frac{1}{2}(y' - nx)^2 + \frac{1}{2}z'^2 - \frac{1}{2}n^2(x^2 + y^2) - U \end{cases}$$

transforms the system (4) into the canonical system

$$(6)\qquad \begin{cases} \dfrac{dx}{dt} = \dfrac{\partial H}{\partial x'}, & \dfrac{dx'}{dt} = -\dfrac{\partial H}{\partial x}, \\ \dfrac{dy}{dt} = \dfrac{\partial H}{\partial y'}, & \dfrac{dy'}{dt} = -\dfrac{\partial H}{\partial y}, \\ \dfrac{dz}{dt} = \dfrac{\partial H}{\partial z'}, & \dfrac{dz'}{dt} = -\dfrac{\partial H}{\partial z}. \end{cases}$$

§ 2. *Integrals of the differential equations.*

Suppose the angular velocity of revolution of the finite bodies, which we now assume move in *circles* according to one or the other of the Lagrangian solutions, is also n; then x_i, y_i, z_i are constants. Therefore, from system (6) there results the integral $H + C = 0$, or, in full,

$$(7)\qquad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = n^2(x^2 + y^2) + \frac{2m_1}{\rho_1} + \frac{2m_2}{\rho_2} + \frac{2m_3}{\rho_3} - C.$$

Let $r_i = \sqrt{(x - x_i)^2 + (y - y_i)^2}$; then, if we suppose that the infinitesimal body moves in the xy-plane, the system of equations (6) reduces to the fourth order, and the integral becomes

$$(8)\qquad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = n^2(x^2 + y^2) + \frac{2m_1}{r_1} + \frac{2m_2}{r_2} + \frac{2m_3}{r_3} - C.$$

If one integral more could be found under this last hypothesis, JACOBI's theory of the *last multiplier* would be applicable, and the problem could be completely solved. However, this system of differential equations is of the type considered by BRUNS in Acta Mathematica, vol. 11, for which he proved that there are no new algebraic integrals. It is equally of the type considered by POINCARÉ in *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. 1, chap. V, for which he proved that there are no new uniform transcendental integrals, even for very small relative masses of two of the three finite bodies.

§ 3. *The Lagrangian solutions when the bodies revolve in circles.*

Suppose first that the three bodies move so as always to lie in a straight line. Suppose the order of the bodies is m_1, m_2, m_3. Suppose the distances between m_1 and m_2, m_2 and m_3, m_3 and m_1 are respectively 1, a_1 and a_2. Therefore $a_2 = 1 + a_1$. Then the equation for the determination of a_1 is *

$$(9)\qquad (m_1 + m_2)a_1^5 + (3m_1 + 2m_2)a_1^4 + (3m_1 + m_2)a_1^3 - (m_2 + 3m_3)a_1^2 - (2m_2 + 3m_3)a_1 - (m_2 + m_3) = 0.$$

* LAPLACE, *Méc. Cél.*, vol. V, p. 310, or TISSERAND, *Méc. Cél.*, vol. I, p. 155.

The a_1 to be chosen is the single real positive root of (9). Then the coördinates of the three bodies are given without ambiguity by

$$(10)\qquad \begin{cases} m_1x_1 + m_2x_2 + m_3x_3 = 0, \\ x_2 - x_1 = 1, \\ x_3 - x_2 = a_1, \\ y_1 = y_2 = y_3 = 0. \end{cases}$$

As there are three arbitrary units, of which only two have been thus far selected, the angular velocity may be chosen arbitrarily, and the unit of mass then determined. Or, if the unit of mass is taken arbitrarily, the angular velocity is given by the equation

$$(11)\qquad n^2 = m_1 + m_2 + m_3\left(\frac{1}{a_2^2} - \frac{1}{a_1^2}\right).$$

There are two other different arrangements of the three bodies upon a straight line, in which their order is cyclically permuted. Their coördinates are determined in a manner similar to that employed above.

In the second Lagrangian solution the three bodies always lie at the vertices of an equilateral triangle. A side of the triangle being taken as the unit of length, the expression for the angular velocity is given by

$$(12)\qquad n^2 = m_1 + m_2 + m_3.$$

Suppose that the axes are so chosen at the origin of time that m_1 is on the x-axis; then the coördinates of the three bodies are determined by the equations

$$(13)\qquad \begin{cases} y_1 = 0, \\ m_1x_1 + m_2x_2 + m_3x_3 = 0, \\ m_1y_1 + m_2y_2 + m_3y_3 = 0, \\ (x_2 - x_1)^2 + (y_2 - y_1)^2 = 1, \\ (x_3 - x_2)^2 + (y_3 - y_2)^2 = 1, \\ (x_1 - x_3)^2 + (y_1 - y_3)^2 = 1. \end{cases}$$

There is evidently but one distinct configuration of this type for three given bodies.

§4. *Surfaces of zero relative velocity in the case of the straight-line Lagrangian solutions.*

The relative velocity of the infinitesimal body being represented by V, equation (7) may be written in this case

$$(14) \qquad V^2 = n^2(x^2+y^2) + \frac{2m_1}{\{(x-x_1)^2+y^2+z^2\}^{\frac{1}{2}}} + \frac{2m_2}{\{(x-x_2)^2+y^2+z^2\}^{\frac{1}{2}}} + \frac{2m_3}{\{(x-x_3)^2+y^2+z^2\}^{\frac{1}{2}}} - C.$$

This equation serves to give the relative velocity of the infinitesimal body at every point of rotating space; or, if the velocity be chosen arbitrarily, it gives the surfaces at which the infinitesimal body will move with the given velocity. The most interesting surfaces are those at which the velocity is zero, for they separate those portions of relative space in which the infinitesimal body may move from those in which it cannot. Setting V equal to zero we have

$$(14)' \qquad n^2(x^2+y^2) + \frac{2m_1}{\{(x-x_1)^2+y^2+z^2\}^{\frac{1}{2}}} + \frac{2m_2}{\{(x-x_2)^2+y^2+z^2\}^{\frac{1}{2}}} + \frac{2m_3}{\{(x-x_3)^2+y^2+z^2\}^{\frac{1}{2}}} = C.$$

The surfaces of which (14)′ is the equation are symmetrical with respect to the xy- and xz-planes, and a line parallel to the z-axis will pierce them in two (or no) points. Moreover, the surfaces are contained within a cylinder whose axis is the z-axis, and whose radius is $\sqrt{C}/n$, to which certain of the folds are asymptotic at $z^2 = \infty$.

The intersection of the surfaces with the xy-plane is given by

$$(15) \qquad n^2(x^2+y^2) + \frac{2m_1}{r_1} + \frac{2m_2}{r_2} + \frac{2m_3}{r_3} = C.$$

For large values of x and y which satisfy this equation we have approximately

$$n^2(x^2+y^2) = C.$$

Therefore for sufficiently large values of C one branch of the curves in the xy-plane is approximately a circle near the asymptotic cylinder.

For small values of x and y satisfying (15) we have

$$\frac{m_1}{r_1} + \frac{m_2}{r_2} + \frac{m_3}{r_3} = \frac{C}{2}.$$

This is the equation of the well-known equipotential curves for three centers of force, and for sufficiently large values of C they consist of closed ovals around each of the bodies. For smaller values of C these ovals coalesce by enlarging, while the large oval mentioned above diminishes in size.

The approximate equations of the curves in the xz-plane, for large and small values respectively of x and z are

$$n^2x^2 = C,$$

$$\frac{m_1}{\{(x-x_1)^2+z^2\}^{\frac{1}{2}}} + \frac{m_2}{\{(x-x_2)^2+z^2\}^{\frac{1}{2}}} + \frac{m_3}{\{(x-x_3)^2+z^2\}^{\frac{1}{2}}} = \frac{C}{2}.$$

The first is the equation of two straight lines parallel to the z-axis; the second is the equation of equipotential curves. Therefore the locus in the xz-plane for sufficiently large values of C is composed of curves asymptotic to the intersection of the asymptotic cylinder with the xz-plane and of ovals around each of the three bodies.

The approximate equations of the curves in yz-plane for large and small values respectively of the coördinates are

$$n^2y^2 = C,$$

$$\frac{m_1}{(x_1^2 + y^2 + z^2)^{\frac{1}{2}}} + \frac{m_2}{(x_2^2 + y^2 + z^2)^{\frac{1}{2}}} + \frac{m_3}{(x_3^2 + y^2 + z^2)^{\frac{1}{2}}} = \frac{C}{2}.$$

The first is the equation of two straight lines parallel to the z-axis. If the bodies lie in the order m_1, m_2, m_3, then x_2 will be much smaller than x_1 and x_3, unless there is a very great difference in the masses m_1 and m_3. If m and m_3 are nearly equal the last equation becomes approximately

$$\frac{m_2}{(x_2^2 + y^2 + z^2)^{\frac{1}{2}}} = \frac{C}{2},$$

which is the equation of a circle. If m_1 is very much greater than m_3, then x_1 is small compared to x_2 and x_3, and we have approximately

$$\frac{m_1}{(x_1^2 + y^2 + z^2)^{\frac{1}{2}}} = \frac{C}{2},$$

which is also the equation of a circle.

Thus we conclude that for sufficiently large values of C, the surfaces are composed of a closed fold, roughly spherical in form, around each of the finite bodies, and curtains hanging from the asymptotic cylinder symmetrically with respect to the xy-plane; for smaller values of C, these folds and curtains unite; and for sufficiently small values of C the surfaces are composed of two parts symmetrical with respect to the xy-plane but not intersecting it.

The motion of the infinitesimal body is real within the small folds around the finite bodies, and outside of the curtain hanging from the asymptotic cylinder, while it is imaginary in all the other portions of relative space. Therefore, if the infinitesimal body is within one of these closed regions at the origin of time, it will forever remain there.

Points on the surfaces may actually be computed most readily by first determining the curves in the xy-plane, and then finding, by methods of approximation, values of z^2 which will satisfy (14)′. This is true since approximate values of z^2 are known from the general nature of the surfaces, when the curves in the xy-plane are given.

If (15) is rationalized and cleared of fractions the resulting equation is of the 40th degree in the variables x and y, and may be used for the determination of points on the curves. The difficulty of locating the roots of an equation of so high degree is supplemented by the presence of foreign solutions which have been introduced in the process of rationalization. The latter difficulty can be avoided entirely and the degree of the equation notably reduced by transforming to bipolar coördinates.

If m_1 is supposed to lie in the negative direction from the origin on the x-axis, it is found without difficulty that

$$x^2 + y^2 = x_2 r_1^2 - x_1 r_2^2 + x_1 x_2 . \tag{16}$$

Then (15) becomes

$$n^2(x_2 r_1^2 - x_1 r_2^2) + \frac{2m_1}{r_1} + \frac{2m_2}{r_2} + \frac{2m_3}{r_3} = C - n^2 x_1 x_2 . \tag{17}$$

r_3^2 can be expressed rationally in terms of r_1 and r_2 by an equation of the second degree. Therefore equation (17) may be written as a polynomial in r_1 and r_2 of the 10th degree. Points on the curves may be found by assuming convenient values of one of the variables and finding the real positive roots of the resulting equation, which will be of the 8th degree in the other variable. The points of intersection of the circles around m_1 and m_2 as centers with the assumed and computed values respectively of r_1 and r_2 as radii, will be points of the locus.

§ 5. *Double points of the surfaces.*

It follows from the symmetrical forms of the surfaces that, as they degenerate and coalesce with decreasing values of C, all of the double points will appear in the xy-plane. Therefore the double points of the curves in the xy-plane alone may be sought. That is, the values of C for which double points occur and their positions must be determined from the equation

$$n^2(x^2 + y^2) + \frac{2m_1}{\{(x - x_1)^2 + y^2\}^{\frac{1}{2}}} + \frac{2m_2}{\{(x - x_2)^2 + y^2\}^{\frac{1}{2}}} + \frac{2m_3}{\{(x - x_3)^2 + y^2\}^{\frac{1}{2}}} = C. \tag{18}$$

The conditions for a double point are

$$\begin{cases} \dfrac{\partial C}{\partial x} \equiv n^2 x - \dfrac{m_1(x - x_1)}{r_1^3} - \dfrac{m_2(x - x_2)}{r_2^3} - \dfrac{m_3(x - x_3)}{r_3^3} = 0, \\ \dfrac{\partial C}{\partial y} \equiv n^2 y - \dfrac{m_1 y}{r_1^3} - \dfrac{m_2 y}{r_2^3} - \dfrac{m_3 y}{r_3^3} = 0. \end{cases} \tag{19}$$

There are certain double points on the x-axis. They are determined by the equation

$$n^2x - \frac{m_1(x-x_1)}{\{(x-x_1)^2\}^{\frac{3}{2}}} - \frac{m_2(x-x_2)}{\{(x-x_2)^2\}^{\frac{3}{2}}} - \frac{m_3(x-x_3)}{\{(x-x_3)^2\}^{\frac{3}{2}}} = 0. \tag{20}$$

The left side of this expression, considered as a function of x, changes sign once, and only once, in each of the intervals $+\infty > x > x_3$, $x_3 > x > x_2$, $x_2 > x > x_1$, $x_1 > x > -\infty$. Therefore there are four double points of the curves on the x-axis, one in each of the intervals mentioned. Expressing equation (20) in terms of r_3 as the variable and clearing of fractions, we have a rational equation of the 7th degree in r_3. It has but one real positive root for the form which it takes in each of the intervals; therefore it locates the positions of the double points, which are on the x-axis, uniquely. By substituting the coördinates of these various points in (18) the corresponding values of C are determined.

The double points for which y is different from zero are found by solving (19) after dividing the second equation by y. By simple combinations of the two we find

$$\begin{cases} (a) \quad n^2x_3 - \dfrac{m_1(x_3-x_1)}{\{(x-x_1)^2+y^2\}^{\frac{3}{2}}} - \dfrac{m_2(x_3-x_2)}{\{(x-x_2)^2+y^2\}^{\frac{3}{2}}} = 0, \\ (b) \quad n^2x_1 + \dfrac{m_2}{\{(x-x_2)^2+y^2\}^{\frac{3}{2}}} + \dfrac{m_3(x_3-x_1)}{\{(x-x_3)^2+y^2\}^{\frac{3}{2}}} = 0, \\ (c) \quad n^2x_2 - \dfrac{m_1}{\{(x-x_1)^2+y^2\}^{\frac{3}{2}}} + \dfrac{m_3(x_3-x_2)}{\{(x-x_3)^2+y^2\}^{\frac{3}{2}}} = 0. \end{cases} \tag{21}$$

Any one of these three equations may be derived from the other two, and the simultaneous solution of any two of them will furnish all of the double points not on the x-axis.

The equations (19) are equally the *necessary* conditions for a minimum value of C. From (18) it is evident that C has a minimum value, and it is easy to show that the *sufficient* conditions have not been fulfilled in the case of any of the solutions for which y is zero, hence it is inferred that there is at least one solution of the system (21).

Suppose $m_1 > m_3$; then $x_1 < 0$, $x_2 > 0$ and $x_3 > x_2 > 0$. The equation (c) is not fulfilled for any value of y^2 when x is greater than x_2; hence there is no solution in the interval $x_2 \leqq x < +\infty$. Likewise (b) is not fulfilled for any value of y^2 when x is less than x_1; hence there is no solution in the interval $x_1 \geqq x > -\infty$. Therefore the solutions must all lie in the interval $x_1 < x < x_2$. For small values of y^2, (b) and (c) are fulfilled by values of x near x_1 and x_2 respectively. For increasing values of y^2 they are fulfilled by values of x continuously increasing in distance from x_1 and x_2 respectively;

therefore there is one, and only one solution in this interval. The *sufficient* conditions for a minimum value of C are fulfilled in this case, and the curves in the xy-plane have degenerated to two points symmetrically situated with respect to the x-axis. There are, therefore, for one distribution of the finite bodies on a line, four double points on the x-axis and two situated symmetrically with respect to it.

In the special case in which $m_1 = m_3$ the solution occurs for $x = x_2 = 0$, the y coördinate being given by the equations

$$(22) \qquad \begin{cases} n^2 = \dfrac{m_1}{4} + m_2, \\ n^2 = \dfrac{m_2}{\{y^2\}^{\frac{3}{2}}} - \dfrac{2m_1}{\{1 + y^2\}^{\frac{3}{2}}} = 0. \end{cases}$$

The y coördinate has a maximum value for $m_2 = 0$, and decreases as m_2 increases.

§6. *Particular solutions of the motion of the infinitesimal body when the finite bodies move in circles according to the straight-line Lagrangian solution.*

The surfaces of zero relative velocity discussed in §4 are surfaces which the infinitesimal body cannot cross. Besides, it is easy to show that the lines of effective force are orthogonal to these surfaces. It follows that if the infinitesimal body is placed at a point in a surface of zero velocity it will move at first orthogonally to the surface. At the double points it seems that there would not be a tendency to move in any direction and it might be inferred that the body would remain relatively at rest.

The direct proof of this is that if the coördinates of the infinitesimal body fulfil the conditions for double points, (19), they also identically verify the differential equations of motion, (4).

There are thus, for each of the three permutations of the order of the finite bodies, four solutions of the problem in which the four bodies all lie in a straight ine, and two in which only the three finite bodies lie in a straight line. Therefore, for three given finite bodies and one of infinitesimal mass, there are eighteen, and no more, different solutions of arbitrary period of revolution, such that the finite bodies lie in a line while the mutual distances of the four bodies are constants.

§ 7. *Surfaces of zero relative velocity in the equilateral triangular Lagrangian solution.*

Suppose the three finite bodies revolve in circles so as to remain at the vertices of an equilateral triangle, and that the axes are chosen so that the x-axis

passes through m_1. Then the equation of the surfaces of zero relative velocity is found from (7) to be

$$n^2(x^2+y^2)+\frac{2m_1}{\{(x-x_1)^2+y^2+z^2\}^{\frac{1}{2}}}+\frac{2m_2}{\{(x-x_2)^2+(y-y_2)^2+z^2\}^{\frac{1}{2}}}+\frac{2m_3}{\{(x-x_3)^2+(y-y_3)^2+z^2\}^{\frac{1}{2}}}=C. \tag{23}$$

The surfaces are symmetrical with respect to the xy-plane, and are enclosed within a cylinder whose axis is the z-axis and whose radius is $\sqrt{C}/n$, and to which certain folds are asymptotic at $z^2=\infty$.

The equation of the trace of the surfaces upon the xy-plane is

$$n^2(x^2+y^2)+\frac{2m_1}{\{(x-x_1)^2+y^2\}^{\frac{1}{2}}}+\frac{2m_2}{\{(x-x_2)^2+(y-y_2)^2\}^{\frac{1}{2}}}+\frac{2m_3}{\{(x-x_3)^2+(y-y_3)^2\}^{\frac{1}{2}}}=C. \tag{24}$$

For respectively large and small values of x and y satisfying this equation we have approximately

$$n^2(x^2+y^2)=C,$$

$$\frac{2m_1}{\{(x-x_1)^2+y^2\}^{\frac{1}{2}}}+\frac{2m_2}{\{(x-x_2)^2+(y-y_2)^2\}^{\frac{1}{2}}}+\frac{2m_3}{\{(x-x_3)^2+(y-y_3)^2\}^{\frac{1}{2}}}=C.$$

The first equation shows that, for sufficiently large values of C, one branch of the curves is approximately a circle lying near the asymptotic cylinder. The second is the equation of the equipotential curves, which, for sufficiently large values of C, consist of closed ovals around each of the three finite bodies. For smaller values of C these smaller ovals coalesce by enlarging, while the large oval unites with the smaller ones by decreasing in size.

The trace of the surfaces in the xz-plane is composed of two curves asymptotic to the lines of intersection of the plane and the asymptotic cylinder, together with an oval around m_1. If the x-axis had been directed so as to pass through either of the other two finite bodies similar results would have been true. Therefore, we conclude that, for sufficiently large values of C, the surfaces are composed of curtains hanging from the asymptotic cylinder, and closed folds approximately spherical in form around each of the finite bodies.

The motion of the infinitesimal body is real within the closed folds and outside of the curtains. If the infinitesimal body were within one of these closed regions at the origin of time it would forever remain there.

In order actually to compute points on the surfaces, it is advantageous to

transform the equation of the curves in the xy-plane to bipolar coördinates. Points on the surfaces can then be rapidly found by methods of approximation.

§ 8. *Double points of the surfaces.*

It follows from the symmetry of the surfaces that the double points which appear as C decreases will all lie in the xy-plane. They will evidently appear, (a) where the small ovals around the finite bodies first touch each other; (b) where the small ovals first touch the large oval; (c) where the small triangular curves, whose sides are formed from two of the small ovals and the large one, vanish from the xy-plane; and (d) where the triangular curves, whose sides are formed from the three small ovals when they unite, vanish from the xy-plane. The points (c) and (d) occur for minima of C as defined by (24), the absolute minimum depending upon the distribution of the masses in the finite bodies.

The conditions for double points are found from (24) to be

$$(25)\qquad \begin{cases} \dfrac{\partial C}{\partial x} \equiv n^2 x - \dfrac{m_1(x-x_1)}{r_1^3} - \dfrac{m_2(x-x_2)}{r_2^3} - \dfrac{m_3(x-x_3)}{r_3^3} = 0, \\ \dfrac{\partial C}{\partial y} \equiv n^2 y - \dfrac{m_1 y}{r_1^3} - \dfrac{m_2(y-y_2)}{r_2^3} - \dfrac{m_3(y-y_3)}{r_3^3} = 0. \end{cases}$$

In order to exhibit most clearly the distribution of the double points and omitting to give the direct proofs from the equations, which would be similar to those given in the former case, let us solve numerically the simple example in which $m_1 = m_2 = m_3 = 1$. Then $n^2 = 3$. Since the figure is symmetrical with respect to the three axes passing through the center of gravity and the three bodies respectively, and since all the double points lie on these axes, it will be sufficient to locate the double points on one of them. In this case the second of (25) is fulfilled for $y = 0$, and the first becomes, a side of the triangle being taken as unit of length,

$$(26)\qquad 3x - \frac{\left(x - \frac{1}{\sqrt{3}}\right)}{\left\{\left(x - \frac{1}{\sqrt{3}}\right)^2\right\}^{\frac{3}{2}}} - \frac{2\left(x + \frac{1}{2\sqrt{3}}\right)}{\left\{\left(x + \frac{1}{2\sqrt{3}}\right)^2 + \frac{1}{4}\right\}^{\frac{3}{2}}} = 0.$$

The real solutions of this equation are approximately $x = 1.22$, $x = 0$, $x = -.24$ and $x = -.93$. The first belongs to the class (b) defined at the beginning of this section, the second to the class (d), the third to the class (a), and the fourth to the class (c). There are corresponding solutions on the other two axes, and these are the only real solutions of the equations. As the point $x = 0$ has been counted three times it is seen that there are ten double points of the surfaces

of zero relative velocity in the case of the triangular solution of the problem of three bodies.

§ 9. *Particular solutions of the motion of the infinitesimal body in case the finite bodies move in circles according to the equilateral triangular Lagrangian solution.*

By general considerations similar to those advanced in § 6 we conclude that the double points are points of zero effective force, and that if the infinitesimal body is placed at one of them with zero relative velocity it will always remain in the same relative position.

The direct proof that these are particular solutions of the problem is that if the relative coördinates of the infinitesimal body are supposed to be constants fulfilling the conditions for double points (25), they also verify identically the differential equations of motion (4).

Therefore, *three finite bodies of arbitrary masses and one infinitesimal body may be started in twenty-eight, and no more, different ways so that they will revolve in an arbitrary period and preserve their mutual distances.*

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DEFINITION OF THE ABELIAN, THE TWO HYPOABELIAN, AND RELATED LINEAR GROUPS AS QUOTIENT-GROUPS OF THE GROUPS OF ISOMORPHISMS OF CERTAIN ELEMENTARY GROUPS*

BY

LEONARD EUGENE DICKSON

1. The present paper aims to give a natural definition of the Abelian and two hypoabelian groups, which moreover preserves the essence of JORDAN's definition based upon his important, but artificial, conception of "exposants d'échange" (*Traité*, pp. 179, 195). A second formal definition, by means of the invariants (11), (14) and (21) below, may be obtained from JORDAN (l. c. p. 217 and pp. 438–440).

Following in the main the developments of JORDAN (l. c. pp. 420–447, in particular) on the construction of solvable groups, we may obtain the above groups as quotient-groups of the groups of isomorphisms of certain elementary groups.† In May, 1898, I communicated such a treatment to Professor MOORE, who emphasized the desirability of presenting the definitions thus obtained for the Abelian and hypoabelian groups in two distinct ways: viz., from the standpoint‡ of JORDAN's linear groups and from the standpoint§ of abstract groups.

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For cross-reference and abstract, cf. Bulletin, April, 1899, pp. 331–2 (where p. 331, line 2 from bottom, for *holoedrically* one should read *meriedrically*).

† For an outline, see §§ 8–10 below. A different aim being in view, the details of the calculation become much simpler than those of JORDAN.

‡ For example, JORDAN (l. c. §§ 118–119) obtains the *non-homogeneous* linear group as the largest group of literal substitutions transforming into itself the commutative literal group readily defined by means of the linear substitutions modulo p,

$$x_i' \equiv x_i + a_i \qquad (i = 1, \cdots, n).$$

§ The idea of the group of isomorphisms of an *abstract* group seems to have been arrived at independently by MOORE and HÖLDER, the former applying it to define the linear *homogeneous* group as the group of transformation into itself of the Abelian group of type $\{1, 1, \cdots, 1\}_n$. See his article in the Bulletin of the American Mathematical Society, Nov., 1895.

I must however refer to the paper by ENRICO BETTI, *Sopra la teorica delle sostituzioni*, Annali di Scienze Matematiche e Fisiche, pp. 5–34, 1855. He obtains a group equivalent to the *homogeneous* linear group as the "massimo Moltiplicatore" of a cyclic group of order

Quite recently I have made the investigation from the latter point of view and have found the method so much simpler and more desirable that I have abandoned my earlier work, which was not fully complete, and give in §§ 8–10 a mere outline of it.

2. Consider the group F generated by the operators of period p (p being prime),

$$\Theta,\ A_i,\ B_i \qquad (i=1,\cdots,m)$$

with the generational relations,*

$$\Theta^p=1,\ A_i^p=1,\ B_i^p=1 \qquad (i=1,\cdots,m) \tag{1}$$

$$\begin{cases} \Theta A_i = A_i\Theta, \quad \Theta B_i = B_i\Theta & (i=1,\cdots,m) \\ A_iA_j = A_jA_i, \quad B_iB_j = B_jB_i & (i,j=1,\cdots,m) \\ A_iB_j = B_jA_i, \quad A_iB_i = \Theta B_iA_i & \left(\begin{smallmatrix} i,j=1,\cdots,m \\ i\neq j \end{smallmatrix}\right). \end{cases} \tag{2}$$

It follows at once that

$$A_i^xB_i^y = \Theta^{xy}B_i^yA_i^x, \tag{3}$$

and hence, in general, if we set

$$S\equiv\Theta^tA_1^{x_1}B_1^{y_1}\cdots A_m^{x_m}B_m^{y_m},$$

$$\Sigma\equiv\Theta^\tau A_1^{\xi_1}B_1^{\eta_1}\cdots A_m^{\xi_m}B_m^{\eta_m},$$

$$S\Sigma = \Theta^{\sum_{i=1}^{m}(x_i\eta_i-\xi_iy_i)}\Sigma S, \tag{4}$$

a relation which includes all of the relations (2).

By application of (2) any operator of F may be put into the form S above. Since A_i is not expressible in terms of the remaining generators, and likewise for B_i, it follows that $S=1$ if and only if

$$x_i\equiv y_i\equiv t\equiv 0 \pmod p \qquad (i=1,\cdots,m)$$

Further, $S=\Sigma$ requires that

$$x_i\equiv\xi_i,\quad y_i\equiv\eta_i,\quad t\equiv\tau \pmod p.$$

Indeed, $S\Sigma^{-1}=1$ may be given the form

$$\Theta^{t-\tau+\sum_{i=1}^{m}\xi_i(y_i-\eta_i)}A_1^{x_1-\xi_1}B_1^{y_1-\eta_1}\cdots A_m^{x_m-\xi_m}B_m^{y_m-\eta_m}=1.$$

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p^p. BETTI observes (p. 34) that this result was given by GALOIS without proof. Although I have not found the passage in the papers of GALOIS referred to by BETTI, nevertheless I am confident that GALOIS had the conception of the group of isomorphisms of the Abelian group of type $\{1,1,\cdots 1\}_n$, at least for $n=2$. See the passage at the bottom of p. 58 of the fragment preserved to us of the posthumous paper, *Des équations primitives qui sont soluble par radicaux*, *Oeuvres Mathématiques D' Évariste Galois*, Paris, 1897.

* The relation $\Theta^p=1$ follows from (3); thus $A_iB_i^p=\Theta^pB_i^pA_i$. We may evidently drop Θ from the list of generators.

It follows from (4) that the powers of Θ are the only operators of F which are commutative with every operator of F.

We may prove by induction the formula

$$S^k = \Theta^{kt-\frac{1}{2}k(k-1)\sum_{i=1}^{m} x_i y_i} A_1^{kx_1} B_1^{ky_1} \cdots A_m^{kx_m} B_m^{ky_m}. \tag{5}$$

Hence, for $p > 2$, every operator of F has the period p and, for $p = 2$, the period 2 or 4, viz.,

$$S^2 = \Theta^{\sum x_i y_i}.$$

3. Every isomorphism I of the group F into itself is obtained by introducing a new set of generators

$$\Theta',\ A_i',\ B_i' \qquad (i = 1, \cdots, m)$$

satisfying the relations (1) and (2) and capable of generating the entire group F. We may set

$$\begin{cases} \Theta' = \Theta^s \\ A_i' = \Theta^{l_i} A_1^{a_{1i}} B_1^{b_{1i}} \cdots A_m^{a_{mi}} B_m^{b_{mi}} \\ B_i' = \Theta^{r_i} A_1^{c_{1i}} B_1^{d_{1i}} \cdots A_m^{c_{mi}} B_m^{d_{mi}} \end{cases} \tag{6}$$

where $s \not\equiv 0 \pmod p$.

Since Θ', A_i', B_i' shall generate F, we must be able, for arbitrarily given values of t', x_i', y_i', to determine t, x_i, y_i so that

$$\Theta^{t'} A_1^{x_1'} B_i^{y_1'} \cdots \equiv \Theta^{t} A_1^{x_1} B_1^{y_1} \cdots \tag{7}$$

Replacing Θ', A_i', B_i' by their values (6) and applying (2) to bring to the left of the product the operators A_1, and next the B_1, etc., we obtain the condition

$$\Theta^{t'} A_1^{x_1'} B_1^{y_1'} \cdots \equiv \Theta^{ts+\sigma} A_1^{\sum_{j=1}^{m} (a_{1j}x_j + c_{1j}y_j)} B_1^{\sum_{j=1}^{m} (b_{1j}x_j + d_{1j}y_j)} \cdots \tag{8}$$

where the exact expression for σ is immaterial. It follows from §2 that the corresponding exponents must be congruent modulo p. Thus t is determined uniquely from

$$t' \equiv ts + \sigma \pmod p.$$

We have further the following $2m$ conditions

$$\begin{cases} x_i' \equiv \sum_{j=1}^{m} (a_{ij}x_j + c_{ij}y_j) \\ y_i' \equiv \sum_{j=1}^{m} (b_{ij}x_j + d_{ij}y_j) \end{cases} \pmod p \qquad (i = 1, \cdots, m) \tag{9}$$

For x_i', y_i' arbitrary, the congruences (9) can be solved for x_i, y_i (and then uniquely) if and only if the determinant

$$D \equiv \begin{vmatrix} a_{11} & c_{11} & \cdot & \cdot \\ b_{11} & d_{11} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} \not\equiv 0 \pmod{p}. \tag{10}$$

We next seek the conditions under which the new generators (6) satisfy the generational relations (2) or the equivalent single relation (4). S denoting the operator (7) and Σ the analogous operator

$$\Sigma \equiv \Theta^{\tau'} A_1^{\xi_1'} B_1^{\eta_1'} \cdots \equiv \Theta'^{\tau} A_1'^{\xi_1} B_1'^{\eta_1} \cdots,$$

the relation (4) requires

$$\Theta^{\sum_{i=1}^{m}(x_i'\eta_i' - \xi_i'y_i')} = \Theta'^{\sum_{i=1}^{m}(x_i\eta_i - \xi_i y_i)}.$$

Since $\Theta' = \Theta^{\tau}$, the substitution (9) and the cogredient substitution on the ξ_i, η_i must have the simultaneous relative invariant

$$\sum_{i=1}^{m}(x_i\eta_i - \xi_i y_i),$$

that is, must satisfy the relation

$$\sum_{i=1}^{m}(x_i'\eta_i' - \xi_i'y_i') \equiv s\sum_{i=1}^{m}(x_i\eta_i - \xi_i y_i). \tag{11}$$

This relation imposes upon the coefficients the following relations modulo p:

$$\begin{cases} \sum_{i=1}^{m}\begin{vmatrix} a_{ij} & c_{ij} \\ b_{ij} & d_{ij} \end{vmatrix} \equiv s, & \sum_{i=1}^{m}\begin{vmatrix} a_{ij} & c_{ik} \\ b_{ij} & d_{ik} \end{vmatrix} \equiv 0, \\ \sum_{i=1}^{m}\begin{vmatrix} a_{ij} & a_{ik} \\ b_{ij} & b_{ik} \end{vmatrix} \equiv 0, & \sum_{i=1}^{m}\begin{vmatrix} c_{ij} & c_{ik} \\ d_{ij} & d_{ik} \end{vmatrix} \equiv 0, \end{cases} \qquad (j, k = 1, \cdots, m;\ j \neq k). \tag{12}$$

From the manner of our derivation of (12) or by direct verification, we observe that (12) form the sufficient (as well as necessary) conditions in order that Θ', A_i', B_i' shall satisfy the relations (2).

It remains to require that A_i' and B_i' have the period p. By the theorem at the end of §2, no new condition is imposed if $p > 2$; while for $p = 2$, we have the conditions

$$\sum_{j=1}^{m} a_{ji}b_{ji} \equiv \sum_{j=1}^{m} c_{ji}d_{ji} \equiv 0 \pmod{2} \qquad (i = 1, \cdots, m). \tag{13}$$

By the same theorem, we find by squaring (7) that the substitution (9) must, for $p = 2$, satisfy the relation

$$\sum_{i=1}^{m} x_i'y_i' \equiv s \cdot \sum_{i=1}^{m} x_i y_i \pmod{2}. \tag{14}$$

Since $s = 1$ for $p = 2$, we find that (14) imposes upon the coefficients of (9) precisely the conditions (12) and (13).

We have now proved the following:

THEOREM.—*The set of generators* (6) *will define an isomorphism I of the group F into itself if, and only if, the exponents satisfy the relations* (10), (12) *and, for $p = 2$, also* (13). *To every isomorphism I corresponds a $2m$-ary linear homogeneous substitution* (9) *belonging to the general Abelian group if $p > 2$ and to the first hypoabelian group if $p = 2$; and inversely, every such substitution* (9) *is so obtainable.*

4. No condition is imposed upon the exponents l_i, r_i in order that (6) define an isomorphism I. Moreover, it is evident that the set of generators

$$(6_1) \qquad \Theta' = \Theta, \quad A_i' = \Theta^{l_i} A_i, \quad B_i' = \Theta^{r_i} B^i \qquad (i = 1, \cdots, m)$$

defines an isomorphism I_1 for which the corresponding substitution (9) is the identity. Inversely, when (9) is the identity, the isomorphism is one of the p^{2m} isomorphisms I_1. Since the group of the I's is isomorphic to the group of the corresponding substitutions (9), the group of the I_1's is self-conjugate under the group of the I's. We have thus the following:

THEOREM.—*The group of isomorphism I of the group F has an invariant sub-group of order p^{2m} formed by the isomorphisms* (6_1). *The quotient-group is simply isomorphic to the general Abelian group, if $p > 2$, and to the first hypoabelian group, if $p = 2$, each on $2m$ indices taken modulo p.*

5. Consider the group F_1 obtained by extending the group F by an operator J commutative with every operator of F and such that $J^p = \Theta$. J is thus of period p^2. Every operator of F_1 may be given the form

$$S \equiv J^t A_1^{x_1} B_1^{y_1} \cdots A_m^{x_m} B_m^{y_m},$$

which reduces to the identity if and only if

$$x_i \equiv y_i \equiv 0 \pmod{p}, \quad t \equiv 0 \pmod{p^2}.$$

On setting

$$\Sigma \equiv J^\tau A_1^{\xi_1} B_1^{\eta_1} \cdots A_m^{\xi_m} B_m^{\eta_m},$$

the relation (4) holds true. Further, $S = \Sigma$ requires that

$$x_1 \equiv \xi_i, \quad y_i \equiv \eta_i \pmod{p}, \quad t \equiv \tau \pmod{p^2}.$$

We readily verify the formula

$$(15) \qquad S^k = J^{kt} \Theta^{-\frac{1}{2}k(k-1)\Sigma x_i y_i} A_1^{kx_1} B_1^{ky_1} \cdots A_m^{kx_m} B_m^{ky_m}.$$

Hence, according as $p > 2$ or $p = 2$, we have

$$(16) \qquad S^p = \Theta^t \text{ or } S^p = \Theta^{t - \Sigma x_i y_i}.$$

By (4) the powers of J are the only operators of F_1 commutative with every operator of F_1. Every isomorphism of F_1 into itself is therefore obtained by introducing new generators of the form

$$(17)\qquad \begin{cases} J' = J^s & (s \text{ not divisible by } p) \\ A_i' = J^{l_i} A_1^{a_{1i}} B_1^{b_{1i}} \cdots A_m^{a_{mi}} B_m^{b_{mi}}, \\ B_i' = J^{r_i} A_1^{c_{1i}} B_1^{d_{1i}} \cdots A_m^{c_{mi}} B_m^{d_{mi}}. \end{cases}$$

As in §3, J', A_i', B_i' will generate the whole group F_1 if, and only if, (10) holds and will satisfy the generational relation (4) if, and only if, the conditions (12) be satisfied. Finally, by (16), A_i' and B_i' will be of period p if, and only if,

$$(18)\qquad \begin{cases} \text{for } p > 2, \quad l_i \equiv r_i \equiv 0 \pmod p & (i = 1, \cdots, m), \\ \text{for } p = 2, \quad l_i - \sum_{j=1}^{m} a_{ji} b_{ji} \equiv r_i - \sum_{j=1}^{m} c_{ji} d_{ji} \equiv 0 \pmod 2 & (i = 1, \cdots, m). \end{cases}$$

For $p > 2$, we set

$$l_i = p\lambda_i, \quad r_i = p\rho_i \qquad (i = 1, \cdots, m).$$

The most general suitable set of the new generators is then

$$\begin{cases} J' = J^s, \ \Theta' = \Theta^s, \\ A_i' = \Theta^{\lambda_i} A_1^{a_{1i}} B_1^{b_{1i}} \cdots, \ B_i' = \Theta^{\rho_i} A_1^{c_{1i}} B_1^{d_{1i}} \cdots. \end{cases}$$

where a_{ji}, b_{ji}, c_{ji}, d_{ji} satisfy (10), (12), while λ_i and ρ_i are arbitrary integers.

For $p = 2$, we set

$$l_i = 2\lambda_i + \sum_{j=1}^{m} a_{ji} b_{ji}, \quad r_i = 2\rho_i + \sum_{j=1}^{m} c_{ji} d_{ji}.$$

The most general set of new generators is then

$$\begin{cases} J' = J^s, \ \Theta' = \Theta^s, \\ A_i' = \Theta^{\lambda_i} J^{\Sigma a_{ji} b_{ji}} A_1^{a_{1i}} B_1^{b_{1i}} \cdots \\ B_i' = \Theta^{\rho_i} J^{\Sigma c_{ji} d_{ji}} A_1^{c_{1i}} B_1^{d_{1i}} \cdots. \end{cases}$$

For $p \equiv 2$, the new set of generators

$$(a)\qquad J' = \Theta^\tau J, \ \Theta' = \Theta, \ A_i' = \Theta^{\lambda_i} A_i, \ B_i' = \Theta^{\rho_i} B_i \qquad (i = 1, \cdots, m)$$

defines an isomorphism for which the corresponding substitution (9) is the identity. Inversely, if the substitution be the identity, the set of generators must be of the above kind. We may state the following:

THEOREM.—*The group of isomorphisms of the group F_1 has an invariant subgroup of order p^{2m+1} formed by the isomorphisms (a). The quotient-group is simply isomorphic with the general Abelian group on $2m$ indices taken modulo p.*

6. Consider the group F' generated by

$$\Theta, A_i, B_i \qquad (i=1, \cdots, m),$$

whose periods are defined by the relations

$$A_1^p = B_1^p = \Theta, \qquad \Theta^p = A_i^p = B_i^p = 1 \qquad (i=2, 3, \cdots, m), \tag{19}$$

and whose further generational relations are given by (2). The general operator of F' may be given the form

$$S \equiv \Theta' A_1^{x_1} B_1^{y_1} \cdots A_m^{y_m} B_m^{y_m},$$

where each exponent may be supposed to be less than p. In this reduced form,* S is the identity if and only if the exponents are all zero.

Two general operators S and Σ of the group F' satisfy the generational relation (4). Formula (5) evidently holds for the group F'. Applying (19) to (5) we obtain for the period of an operator S of F' the formulæ

$$\begin{cases} S^p = \Theta^{x_1 + y_1} & (\text{if } p > 2), \\ S^p = \Theta^{x_1 + y_1 - \sum_{i=1}^{m} x_i y_i} & (\text{if } p = 2). \end{cases} \tag{20}$$

To obtain the group of isomorphisms of F', we proceed as in § 3, the only variation being in regard to the periods of A_i', B_i'. Raising the identity (7) to the p^{th} power and applying (20), we obtain the periodicity conditions,

$$\Theta^{x_1' + y_1'} = \Theta^{\prime x_1 + y_1} \qquad (\text{if } p > 2).$$

$$\Theta^{x_1' + y_1' - \Sigma x_i' y_i'} = \Theta^{\prime x_1 + y_1 - \Sigma x_i y_i} \qquad (\text{if } p = 2),$$

For $p = 2$, $\Theta' \equiv \Theta' = \Theta$, so that the condition is

$$x_1' + y_1' + \sum_{i=1}^{m} x_i' y_i' \equiv x_1 + y_1 + \sum_{i=1}^{m} x_i y_i \pmod 2, \tag{21}$$

which imposes upon the coefficients of the substitution (9) the conditions (12) together with the following conditions modulo 2 (where $\delta_{11} = 1$, $\delta_{12} = \cdots = \delta_{1m} = 0$),†

$$a_{1j} + b_{1j} + \sum_{i=1}^{m} a_{ij} b_{ij} \equiv c_{1j} + d_{1j} + \sum_{i=1}^{m} c_{ij} d_{ij} \equiv \delta_{1j} \qquad (j=1, \cdots, m). \tag{22}$$

As one sees directly from (20), the conditions (22) suffice when $p = 2$ to give A_i' and B_i' the periods (19). For $p = 2$, the group of substitutions (9) having the invariant (21) is the second hypoabelian group. Proceeding as in § 4, we have the following result:

* Unless so reduced, $S \equiv 1$ if and only if

$$x_i \equiv y_i \equiv t \equiv 0 \pmod p, \qquad x_1 + y_1 + pt \equiv 0 \pmod{p^2}$$

† The relation (21) is not a formal, but a numerical, identity. Indeed, we must note that $x^2 \equiv x \pmod 2$.

THEOREM.—*The group of isomorphisms of the group F' has an invariant subgroup of order p^{2m} formed by the isomorphisms* (6_1). *Their quotient-group is simply isomorphic, for $p = 2$, to the second hypoabelian group, and, for $p > 2$, to that subgroup of the general Abelian group satisfying the relation*

$$x_1' + y_1' \equiv s\,(x_1 + y_1)\,, \tag{23}$$

where s is any integer not divisible by p, but the same as in (12).

Since the Abelian group contains the substitution L_1, which alters only x_1, replacing it by $x_1 + y_1$, the subgroup defined by (23) is conjugate within the general Abelian group to that subgroup which multiplies x_1 by the parameter s.

7. Consider, finally, the group F_1' obtained by extending F' by an operator J commutative with every operator of F' and such that $J^p = \Theta$. To obtain the group of isomorphisms of F_1', we proceed as in § 5. The conditions that the new generators (17) shall have the periods (19) are seen, on applying (15), to be as follows:

$$\text{(for } p > 2) \qquad l_i + a_{1i} + b_{1i} \equiv r_i + c_{1i} + d_{1i} \equiv \delta_{1i}$$

$$\text{(for } p = 2) \quad l_i + a_{1i} + b_{1i} - \sum_{j=1}^{m} a_{ji} b_{ji} \equiv r_i + c_{1i} + d_{1i} - \sum_{j=1}^{m} c_{ji} d_{ji} \equiv \delta_{1i}$$

$$(i = 1, \cdots, m).$$

The exponents l_i, r_i are thus determined modulo p. Proceeding as in § 5, we reach the following result:

THEOREM.—*The group of isomorphisms of F_1' has an invariant subgroup of order p^{2m+1}, the quotient-group being simply isomorphic with the general Abelian group on $2m$ indices modulo p.*

8. We may give a concrete representation * of the operators of the groups F, F_1, F', F_1', as linear homogeneous substitutions on p^m variables $Z_{\xi_1, \cdots, \xi_m}$ the indices ξ_i being taken modulo p.

For the groups F and F_1 we may take

$$\begin{cases} A_i: & Z'_{\xi_1 \cdots \xi_m} = \theta^{\xi_i} Z_{\xi_1 \cdots \xi_m}, \\ B_i: & Z'_{\xi_1 \cdots \xi_i \cdots \xi_m} = Z_{\xi_1 \cdots \xi_i + 1 \cdots \xi_m}, \\ \Theta: & Z'_{\xi_1 \cdots \xi_m} = \theta Z_{\xi_1 \cdots \xi_m}, \\ J: & Z'_{\xi_1 \cdots \xi_m} = j Z_{\xi_1 \cdots \xi_m}, \end{cases} \tag{24}$$

where θ is a primitive p^{th} root of unity and j a primitive root (always existing) of the equation

$$j^p = \theta \quad \text{or} \quad j^{p^2} = 1.$$

It is readily seen that the substitutions (24) satisfy the relations (1) and (2).

* Taken in a simplified form from JORDAN'S *Traité*, §§ 561 and 566. For JORDAN'S π, I use p, and avoid the use of his other parameter p.

The general substitution S of F takes the form

$$Z'_{\xi_1 \cdots \xi_m} = \theta^{t+\sum_{i=1}^{m} x_i(\xi_i+y_i)} Z_{\xi_1+y_1 \cdots \xi_m+y_m}. \tag{25}$$

It is at once evident that two substitutions S are identical if and only if their corresponding exponents x_i, y_i, t are congruent modulo p.

9. By a method similar to that of JORDAN, l. c. §§ 568–9, but simpler as to details, we can readily prove that the characteristic determinant (with the parameter K) of (25) is equal modulo p to respectively

$$(1-K^p)^{p^m-1} \qquad (p>2),$$

$$(K^2-\theta^{\Sigma x_i y_i})^{2^m-1} \qquad (p=2).$$

Since the substitution S and its transformed (7) by a linear substitution must have equal characteristic determinants, we obtain (14) as a condition *necessary* for the isomorphism when $p=2$. For the group of transformation of F_1 into itself, the characteristic determinant of

$$J^t A_1'^{x_1} B_1'^{y_1} \cdots \equiv J^{\sum_{i=1}^{m}(l_i x_i + r_i y_i)+t} \Theta^{\tau} A_1^{x_1'} B_1^{y_1'} \cdots$$

is, for $p=2$,

$$\left(K^2-\theta^{t+\sum_{i=1}^{m}(l_i x_i + r_i y_i + x_i' y_i')}\right)^{2^m-1}.$$

Hence a condition necessary for the isomorphism is the following,

$$\sum_{i=1}^{m}(x_i' y_i' + l_i x_i + r_i y_i) \equiv \sum_{i=1}^{m} x_i y_i \pmod 2,$$

whence follow the relations $(18)_{p=2}$.

But by this method it remains to discuss the following inverse problem. How many isomorphisms of F (or F_1) into itself correspond to each substitution (9) satisfying the above relations? Compare JORDAN, l. c. §§ 579–580.

10. For the groups F' and F_1' of §§ 6 and 7, we may take

$$\begin{cases} A_1: & Z'_{\xi_1 \cdots \xi_m} = j\theta^{\xi_1} Z_{\xi_1 \cdots \xi_m} \\ B_1: & Z'_{\xi_1 \xi_2 \cdots \xi_m} = j Z_{\xi_1+1\, \xi_2 \cdots \xi_m} \\ \Theta, J, A_i, B_i\ (i=2, \cdots, m) & \text{defined by (24).} \end{cases}$$

Then the general substitution $\Theta_1'^{t} A^{x_1} B^{y_1} \cdots$ is

$$Z'_{\xi_1 \cdots \xi_m} = j^{x_1+y_1} \theta^{t+\Sigma x(\xi_i+y_i)} Z_{\xi_1+y_1 \cdots \xi_m+y_m},$$

whose characteristic determinant is for $p=2$,

$$(K^2-\theta^{x_1+y_1+\Sigma x_i y_i})^{2^m-1}.$$

This equals the characteristic determinant of

$$\Theta'^{t} A_1'^{x_1} B_1'^{y_1} \cdots A_m'^{x_m} B_m'^{y_m}$$

only if (21) be satisfied.

NOTE ON THE UNILATERAL SURFACE OF MOEBIUS*

BY

HEINRICH MASCHKE

In order to construct an algebraic surface containing as a part the unilateral paper-strip of MOEBIUS,† let a straight line L move in space along a circle C, perpendicular to the tangents of C and in such a way that, when the point of intersection Q of L with C has described the full circle, the initial position of L makes with its final position an angle of 180°. The condition that L meets C at right angles is equivalent to the condition that L meets a straight line A passing through the center M of the circle and perpendicular to its plane; let P be the movable point of intersection of L and A. If now we add the further condition that the range P on A be projective to the range Q on C (*e. g.*, by taking the angle QPM always half the angle of the arc described by Q on C) then L describes, according to a general theorem,‡ a *ruled surface of the third order.*

Conversely: take any ruled surface R of the third order, particular cases excepted, pass a plane section through one of the generators L which will meet R besides L in a conic section K, and describe a curve T on R the points of which have along the generators a sufficiently small constant distance from K; then T will cut out of R a unilateral Moebius surface.

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† MOEBIUS: *Ueber die Bestimmung des Inhalts eines Polyeders*, § 11. (Gesammelte Werke, vol. II, p. 484–485.)

‡ REYE: *Geometrie der Lage*, ed. 3, vol. I, p. 209, no. 101.

ON REGULAR SINGULAR POINTS OF LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER WHOSE COEFFICIENTS ARE NOT NECESSARILY ANALYTIC*

BY

MAXIME BÔCHER

§1. *Introduction.*

Ever since the time of CAUCHY it has been considered of interest to establish the existence of solutions of differential equations whose coefficients are functions of a real variable x, and to do this without requiring these coefficients to be analytic functions of x, but merely continuous functions. It is a natural extension of this point of view to wish to investigate the nature of singular points of such equations, *i. e.*, of points where the coefficients become discontinuous.† It is my object in the present paper to carry through such an investigation in a special, but at the same time important, case.

We shall confine ourselves to the equation:

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0, \tag{1}$$

and shall, as has already been said, regard the independent variable x as real, while p and q are functions of x which are not required to be analytic. In fact, since no further complication is thereby introduced, we shall not even require that p and q be real. A point at which p or q is discontinuous we call a singular point of (1), and we shall restrict ourselves to singular points which, at least on one side, are isolated from all other singular points; *i. e.*, if a is such a point, it must be possible to find a positive constant ϵ so small that *either* in the interval $a < x < a + \epsilon$, *or* in the interval $a > x > a - \epsilon$ there lies no singular point of the differential equation. In order to simplify matters we will take our singular point at the point $x = 0$, and will suppose that throughout the interval $0 < x \leqq b$, b being therefore positive, there lies no singular point of (1). We have, of course, hereby introduced no essential specializacion into our problem.

* Presented to the Society Dec. 28, 1899. Received for publication Jan. 2, 1900.

† Four papers by Kneser (CRELLE, vols. 116, 117, 120 and MATH. ANN., vol. 49) may be regarded as a contribution to this subject, the singular point being at infinity.

Finally we will assume that we can write:

$$p = \frac{\mu}{x} + p_1, \quad q = \frac{\nu}{x^2} + q_1$$

where μ and ν are constants, and where:

$$\int_b^\epsilon |p_1|\, dx \quad \text{and} \quad \int_b^\epsilon x\, |q_1|\, dx$$

approach finite limits as the positive quantity ϵ approaches zero.

When all these conditions are fulfilled we will speak of the point $x = 0$ as a *regular singular point.*

I have already investigated such singular points, to some extent, in a paper presented to the Society in October 1898 and printed in the Bulletin in March 1899. The method which I then used is not sufficiently general to treat more than part of the problem in hand. In the following pages I have adopted a more direct method which enables me to treat all cases of the problem in question. This method also admits of application to linear differential equations of the n^{th} order, as I hope to show on a future occasion. I hope also soon to present to the Society certain applications of the results contained in the present paper.

§ 2. *The Larger Exponent.*

We will now proceed to develop solutions of (1) about the point $x = 0$ into series valid throughout at least that portion of the interval $0\,b$ which lies sufficiently near to $x = 0$. For this purpose we use a method of successive approximations.

We write (1) in the form:

$$\frac{d^2y}{dx^2} + \frac{\mu}{x}\frac{dy}{dx} + \frac{\nu}{x^2}\, y = -\, p_1 \frac{dy}{dx} - q_1 y\,.$$

Starting from the value $y = 0$, we substitute this value in the second member and get the following equation for computing a first approximation Y_0:

$$\frac{d^2y}{dx^2} + \frac{\mu}{x}\frac{dy}{dx} + \frac{\nu}{x^2}\, y = 0\,. \tag{2}$$

This equation has the solutions:

$$\eta_1 = x^{\kappa'} \quad \eta_2 = x^{\kappa''}$$

where κ' and κ'' are the roots of the quadratic equation in ρ:

$$\rho(\rho - 1) + \mu\rho + \nu = 0\,.$$

This equation we will call the *indicial equation* of the point $x = 0$ and κ', κ'' we will call the *exponents* of this point. Their difference $\kappa' - \kappa''$ we will denote by κ. For the present we exclude the case in which $\kappa' = \kappa''$. We will suppose that our notation has been so chosen that

$$R\kappa' \geqq R\kappa'',$$

where the symbol R stands for the words "the real part of."

We begin by taking as our first approximation:

$$Y_0 = \eta_1 = x^{\kappa'}.$$

Then we compute in succession the quantities $Y_1, Y_2, \cdots$ from the equations:

$$(3) \qquad \frac{d^2Y_m}{dx^2} + \frac{\mu}{x}\frac{dY_m}{dx} + \frac{\nu}{x^2}Y_m = -p_1\frac{dY_{m-1}}{dx} - q_1 Y_{m-1} \qquad (m = 1, 2, \cdots),$$

the constants that come in through the integration of these differential equations being so determined that

$$Y_m = x^{\kappa'}\varphi_m(x), \text{ where } \lim_{x=0}\varphi_m(x) = 0.$$

That such a determination of these constants of integration is possible will appear in the course of our work.

The equation (3) is a non-homogeneous linear differential equation for determining Y_m. We solve it by the method of variation of constants, *i. e.*, we let:

$$Y_m = f_m^{(1)}\eta_1 + f_m^{(2)}\eta_2$$

and determine $f_m^{(1)}$ and $f_m^{(2)}$ so that not only the above expression satisfies (3) but also:

$$Y_m' = f_m^{(1)}\eta_1' + f_m^{(2)}\eta_2',$$

(accents here and in what follows denoting differentiation). This gives for $f_m^{(1)}$ and $f_m^{(2)}$ the values:

$$(4) \qquad f_m^{(1)} = -\int\frac{\eta_2 P_m}{\eta_1\eta_2' - \eta_2\eta_1'}dx, \qquad f_m^{(2)} = \int\frac{\eta_1 P_m}{\eta_1\eta_2' - \eta_2\eta_1'}dx,$$

where, for the sake of abbreviation, we have denoted the second member of (3) by P_m. The constants involved in these indefinite integrals we determine by writing them as definite integrals with upper limit x and lower limit 0. This must be shown to be allowable by proving that these definite integrals converge.

As we have already said, we shall not consider the whole interval $0b$ but only a portion $0c$ of it, which is presently to be chosen so short that certain conditions

are satisfied. We will introduce the positive constant C by the following inequalities:

$$C > |\kappa'|\,, \quad C > |\kappa''|\,, \quad C > 1\,.$$

Further let:

$$M(x) = \int_0^x \{C|p_1| + x|q_1|\}\,dx\,.$$

It should be noticed that $M(x) \leqq M(c)$ when $0 \leqq x \leqq c$.

We will now establish the convergence of the integrals (4), which may at once be reduced to:

$$(4') \qquad \begin{aligned} f_m^{(1)} &= \frac{1}{\kappa}\int_0^x x^{1-\kappa'} P_m\,dx\,, \\ f_m^{(2)} &= \frac{-1}{\kappa}\int_0^x x^{1-\kappa''} P_m\,dx\,, \end{aligned} \qquad (m = 1,\ 2,\ 3,\ \cdots)$$

and at the same time prove that Y_m satisfies the inequalities:

$$(5) \qquad \begin{aligned} |Y_m(x)| &\leqq \frac{C^m\{2M(x)\}^m}{|\kappa|^m}\, x^{R\kappa'}\,, \\ |Y_m'(x)| &\leqq \frac{C^{m+1}\{2M(x)\}^m}{|\kappa|^m}\, x^{R\kappa'-1}\,. \end{aligned} \qquad (m = 0,\ 1,\ 2,\ \cdots)$$

The inequalities (5) obviously hold when $m = 0$. Let us then assume that when $m = 0,\ 1,\ \cdots\ (m_1 - 1)$, the inequalities (5) hold and the integrals (4') converge. Then, using the ordinary method of mathematical induction, we have merely to prove that when $m = m_1$ the integrals (4') converge and formulæ (5) hold. Now we have:

$$P_{m_1} = -\,p_1 Y'_{m_1-1} - q_1 Y_{m_1-1}\,;$$

therefore:

$$|P_{m_1}| \leqq |p_1|\,|Y'_{m_1-1}| + |q_1|\,|Y_{m_1-1}| \leqq \frac{C^{m_1-1}\{2M(x)\}^{m_1-1}}{|\kappa|^{m_1-1}}\, x^{R\kappa'-1}\{C|p_1| + x|q_1|\}\,,$$

from which we see not merely that the integrals (4') converge when $m = m_1$, but that

$$|f_{m_1}^{(1)}| \leqq \frac{1}{|\kappa|}\int_0^x x^{1-R\kappa'}|P_{m_1}|\,dx \leqq \frac{C^{m_1-1}\{2M(x)\}^{m_1-1}M(x)}{|\kappa|^{m_1}}$$

$$|f_{m_1}^{(2)}| \leqq \frac{1}{|\kappa|}\int_0^x x^{1-R\kappa''}|P_{m_1}|\,dx \leqq \frac{C^{m_1-1}\{2M(x)\}^{m_1-1}M(x)}{|\kappa|^{m_1}}\, x^{R\kappa}.$$

Therefore:

$$|Y_{m_1}| \leqq |f_{m_1}^{(1)}|\,|\eta_1| + |f_{m_1}^{(2)}|\,|\eta_2| \leqq \frac{C^{m_1-1}\{2M(x)\}^{m_1}}{|\kappa|^{m_1}}\, x^{R\kappa'},$$

$$|Y'_{m_1}| \leqq |f_{m_1}^{(1)}|\,|\eta'_1| + |f_{m_1}^{(2)}|\,|\eta'_2| \leqq \frac{C^{m_1}\{2M(x)\}^{m_1}}{|\kappa|^{m_1}}\, x^{R\kappa'-1}.$$

From these inequalities formulæ (5), for the case $m = m_1$, follow at once when we remember that $C > 1$.

By means of (5) we not only verify a statement made above, but deduce another fact not yet stated. Let:

$$Y_m = x^{\kappa'}\varphi_m(x), \qquad Y'_m = x^{\kappa'-1}\bar{\varphi}_m(x) \qquad (m = 0, 1, 2, \cdots).$$

From (5) we get immediately:

$$(5) \qquad |\varphi_m(x)| \leqq \frac{C^m\{2M(x)\}^m}{|\kappa|^m}, \qquad |\bar{\varphi}_m(x)| \leqq \frac{C^{m+1}\{2M(x)\}^m}{|\kappa|^m},$$

these inequalities holding throughout the interval $0 < x \leqq c$. Since $M(x)$ approaches zero with x the same will be true of $\varphi_m(x)$ and $\bar{\varphi}_m(x)$ when $m > 0$. We will therefore define: $\varphi_m(0) = \bar{\varphi}_m(0) = 0$ $(m = 1, 2, 3, \cdots)$.

The absolute and uniform convergence of the two series:

$$(6) \qquad \sum_{m=0}^{m=\infty}\varphi_m(x), \qquad \sum_{m=0}^{m=\infty}\bar{\varphi}_m(x),$$

throughout the interval $0 \leqq x \leqq c$ now follows at once from (5), by means of the fundamental Weierstrassian criterion,* provided that:

$$\frac{2CM(c)}{|\kappa|} < 1.$$

We will, from now on, suppose that c is so small that this inequality is true.

By multiplying the series (6) by $x^{\kappa'}$ and $x^{\kappa'-1}$ respectively we get the series:

$$\sum_{m=0}^{m=\infty} Y_m, \qquad \sum_{m=0}^{m=\infty} Y'_m.$$

From the uniform convergence of the series (6) throughout the interval $0 \leqq x \leqq c$ we cannot infer the uniform convergence of the last written series throughout this interval, since the factors $x^{\kappa'}$ and $x^{\kappa'-1}$ may become infinite at $x = 0$. We can however infer at once their uniform convergence throughout the interval $\epsilon \leqq x \leqq c$ where ϵ is any positive constant less than c.

* WEIERSTRASS, Werke, vol. II, p. 202.

Since the second of the two series just written is uniformly convergent throughout the interval $\epsilon \leqq x \leqq c$, and since its terms are the derivatives of the terms of the first, it follows that it represents the derivative of the function represented by the first series at every point of the interval $\epsilon \leqq x \leqq c$, and therefore at every point of the interval $0 < x \leqq c$.

We will write:

$$y_1 = \sum_{m=0}^{m=\infty} Y_m, \qquad y_1' = \sum_{m=0}^{m=\infty} Y_m'.$$

We wish to prove that y_1 satisfies (1) at every point of the interval $0 < x \leqq c$. Consider any point x of this interval and take a positive constant ϵ so small that $\epsilon < x \leqq c$. Throughout this last interval each one of the following four series is obviously uniformly convergent:

$$\begin{aligned}
\frac{\nu}{x^2} y_1 &= \frac{\nu}{x^2} Y_0 + \frac{\nu}{x^2} Y_1 + \frac{\nu}{x^2} Y_2 + \cdots, \\
q_1 y_1 &= \qquad q_1 Y_0 + q_1 Y_1 + \cdots, \\
\frac{\mu}{x} y_1' &= \frac{\mu}{x} Y_0' + \frac{\mu}{x} Y_1' + \frac{\mu}{x} Y_2' + \cdots, \\
p_1 y_1' &= \qquad p_1 Y_0' + p_1 Y_1' + \cdots.
\end{aligned}$$

Adding these four equations we get by (3):

$$\left(\frac{\mu}{x} + p_1\right) y_1' + \left(\frac{\nu}{x^2} + q_1\right) y_1 = -Y_0'' - Y_1'' - Y_2'' - Y_3'' \cdots.$$

The series on the right must be uniformly convergent, since it is the sum of four uniformly convergent series. It therefore represents the function $-y_1''$, and we thus see that y_1 satisfies (1) at the point x which is any point of the interval $0 < x \leqq c$.

The results we have so far obtained may be summed up (in a slightly different form from that in which we have yet stated them) as follows:

A solution y_1 of the differential equation (1) *and its derivative y_1' may be developed about the point $x = 0$ as follows:*

$$\begin{aligned}
y_1 &= x^{\kappa'}[1 + \varphi_1(x) + \varphi_2(x) + \cdots], \\
y_1' &= x^{\kappa'-1}[\kappa' + \bar{\varphi}_1(x) + \bar{\varphi}_2(x) + \cdots],
\end{aligned}$$

these developments being valid throughout a certain interval $0 < x \leqq c$. Here the series in parenthesis converge uniformly and absolutely throughout the interval $0 \leqq x \leqq c$, and the functions φ and $\bar{\varphi}$ are continuous throughout this

last mentioned interval, and vanish when $x = 0$. *Except at the point* $x = 0$ *these functions are determined by the recurrent formulæ:*

$$\varphi_m(x) = \frac{1}{\kappa}\Big[\int_0^x Q_m(x)dx - x^{-\kappa}\int_0^x x^{\kappa} Q_m(x)dx\Big],$$

$$\bar{\varphi}_m(x) = \frac{1}{\kappa}\Big[\kappa'\int_0^x Q_m(x)dx - \kappa'' x^{-\kappa}\int_0^x x^{\kappa} Q_m(x)dx\Big],$$

where

$$Q_m(x) = -p_1\bar{\varphi}_{m-1}(x) - xq_1\varphi_{m-1}(x), \qquad (m = 2, 3, \cdots)$$

and

$$Q_1(x) = -p_1\kappa' - xq_1.$$

§ 3. *The Smaller Exponent.*

The solution y_1 found in the last section may be written:

$$y_1 = x^{\kappa'}E_1(x),$$

where $E_1(x)$ is continuous throughout the interval $0 \leqq x \leqq c$, and $E_1(0) = 1$. We will take c so small that $E_1(x)$ does not vanish when $0 \leqq x \leqq c$.

A second solution is given by the well known formula:

$$y_2 = y_1\int \frac{e^{-\int p dx}}{y_1^2}\,dx.$$

The constants of integration here involved may be explicitly written as follows:

$$y_2 = ky_1\int_c^x \frac{x^{-\mu}e^{-\int_0^x p_1 dx}}{y_1^2}\,dx + Cy_1$$

$$= kx^{\kappa'}E_1(x)\int_c^x x^{-\kappa-1}\frac{e^{-\int_0^x p_1 dx}}{\{E_1(x)\}^2}\,dx + Cy_1.$$

Let us write:

$$\frac{e^{-\int_0^x p_1 dx}}{\{E_1(x)\}^2} = 1 + F(x).$$

It will be noticed that $F(x)$ is continuous throughout the interval $0 \leqq x \leqq c$, and that $F(0) = 0$. We may now write:

$$y_2 = -\frac{k}{\kappa}x^{\kappa''}E_1(x) + \frac{k}{\kappa}c^{-\kappa}x^{\kappa'}E_1(x) + kx^{\kappa'}E_1(x)\int_c^x x^{-\kappa-1}F(x)dx + Cy_1.$$

In order now to get as simple a function y_2 as possible, we will choose the as yet undetermined constants k and C as follows:

$$k = -\kappa, \quad C = c^{-\kappa}.$$

Then:

$$y_2 = x^{\kappa''} E_1(x)\left\{1 - \kappa x^{\kappa} \int_c^x x^{-\kappa-1} F(x)dx\right\}.$$

The second term which stands here in brackets is evidently continuous when $0 < x \leqq c$, and approaches 0 with x.* We may therefore write:

$$y_2 = x^{\kappa''} E_2(x)$$

where $E_2(x)$ is continuous throughout the interval $0 \leqq x \leqq c$, and $E_2(0) = 1$.

In the last section we obtained for the derivative of y_1 a value which may be written:

$$y_1' = x^{\kappa'-1} H_1(x)$$

where $H_1(x)$ is continuous throughout the interval $0 \leqq x \leqq c$ and $H_1(0) = \kappa'$. We can get a similar expression for y_2'; for we may write:

$$y_2 = x^{-\kappa} y_1 - \kappa x^{\kappa'} \int_c^x x^{-\kappa-1} F(x)dx.$$

Differentiating this and replacing y_1 and y_1' by the values $x^{\kappa'} E_1(x)$ and $x^{\kappa'-1} H_1(x)$ respectively we get:

$$y_2' = -\kappa x^{\kappa''-1} E_1(x) + x^{\kappa''-1} H_1(x) - \kappa x^{\kappa''-1} F(x) - x^{\kappa''-1}\left(\kappa\kappa' x^{\kappa} \int_c^x x^{-\kappa-1} F(x)dx\right).$$

And this finally may be written:

$$y_2' = x^{\kappa''-1} H_2(x),$$

where $H_2(x)$ is continuous throughout the interval $0 \leqq x \leqq c$ and $H_2(0) = \kappa''$.

Summing up the principal results of this section and the last we have the theorem:

If the exponents κ' and κ'' are unequal, the equation (1) *always has two solutions y_1 and y_2 which together with their derivatives may be written:*

$$y_1 = x^{\kappa'} E_1(x), \quad y_2 = x^{\kappa''} E_2(x), \quad y_1' = x^{\kappa'-1} H_1(x), \quad y_2' = x^{\kappa''-1} H_2(x).$$

where $E_1(x)$, $E_2(x)$, $H_1(x)$, $H_2(x)$ are continuous throughout the interval $0 \leqq x \leqq c$, and $E_1(0) = E_2(0) = 1$, $H_1(0) = \kappa'$, $H_2(0) = \kappa''$.

The case in which the coefficients p and q of (1) are real deserves special mention. Here μ and ν are real, and therefore κ' and κ'' are either real or conjugate imaginary. In the first case it is clear that all four functions E_1, E_2, H_1, H_2 are real.

* A proof of this will be found in the Bulletin for October, 1898, p. 25.

The case in which κ' and κ'' are conjugate imaginary is particularly interesting because then $R\kappa' = R\kappa''$ and we can therefore get *both* solutions y_1 and y_2 by the method of successive approximations of §2. It is readily seen that in this case the functions E_1 and E_2 are conjugate imaginary functions:

$$E_1(x) = E(x) + iF(x), \quad E_2(x) = E(x) - iF(x).$$

Here E and F are continuous and real throughout the interval $0 \leqq x \leqq c$ and $E(0) = 1$, $F(0) = 0$.

The two solutions y_1 and y_2 will now be complex. The expressions:

$$y_3 = \frac{y_1 + y_2}{2}, \quad y_4 = \frac{y_1 - y_2}{2i}$$

give us however real solutions, which, if we let:

$$\kappa' = \alpha + \beta i, \ \kappa'' = \alpha - \beta i$$

have the values:

$$y_3 = x^\alpha\{\cos(\beta \log x)E(x) - \sin(\beta \log x)F(x)\},$$
$$y_4 = x^\alpha\{\sin(\beta \log x)E(x) + \cos(\beta \log x)F(x)\}.$$

§ 4. *The Case of Equal Exponents.*

We come now to the case, which has so far been excluded, in which $\kappa' = \kappa''$. We shall find here that the assumption that $|p_1|$ and $x|q_1|$ can be integrated up to the point $x = 0$ is no longer sufficient. *We will therefore now assume that*

$$\int_b^\epsilon |p_1|(\log x)^2 dx, \qquad \int_b^\epsilon x|q_1|(\log x)^2 dx$$

approach finite limits as ϵ approaches zero.

In order to get two independent solutions of (2) we must now write:

$$\eta_1 = x^{\kappa'}, \qquad \eta_2 = x^{\kappa'} \log x.$$

Either of these two functions may be used as our first approximation. We will consider these two cases in succession:

(*a*) *We start from the function* $Y_{0\,1} = \eta_1$, and compute the functions $Y_{1\,1}$, $Y_{2\,1}$, $Y_{3\,1}$, $\cdots$ by means of (3). This gives as before:

$$Y_{m\,1} = f_{m\,1}^{(1)}\eta_1 + f_{m\,1}^{(2)}\eta_2, \qquad Y'_{m\,1} = f_{m\,1}^{(1)}\eta'_1 + f_{m\,1}^{(2)}\eta'_2,$$

where $f_{m\,1}^{(1)}$ and $f_{m\,1}^{(2)}$ are given by formulæ (4) which now reduce not to (4′) but to:

$$(4'') \qquad \begin{aligned} f_{m\,1}^{(1)} &= -\int_0^x x^{1-\kappa'} \log x \, P_m dx, \\ f_{m\,1}^{(2)} &= \int_0^x x^{1-\kappa'} P_m dx. \end{aligned}$$

Let us now restrict the length of the interval $0c$ by requiring that $1/c \geqq e$ (the exponential base). It follows that:

$$|\log x| \geqq 1 \qquad (0 < x \leqq c).$$

We further introduce the two quantities C and M:

$$C > |\kappa'|, \quad C > 1, \quad M(x) = \int_0^x \{ C|p_1| + x|q_1| \} |\log x| \, dx.$$

We now establish the convergence of the integrals $(4'')$ and at the same time prove that Y_{m1} satisfies the following inequalities:

$$(5') \qquad \begin{aligned} |Y_{m1}| &\leqq \{3M(x)\}^m C^m x^{R\kappa'}, \\ |Y'_{m1}| &\leqq \{3M(x)\}^m C^{m+1} x^{R\kappa'-1}. \end{aligned}$$

These formulæ evidently hold when $m = 0$. Let us assume that the integrals $(4'')$ converge and that formulæ $(5')$ hold when $m \leqq m_1 - 1$. It remains merely to prove that the same is true when $m = m_1$. We have:

$$|P_{m_1}| \leqq |p_1| \, |Y'_{m_1-1\,1}| + |q_1| \, |Y_{m_1-1\,1}| \leqq \{3M(x)\}^{m_1-1} C^{m_1-1} x^{R\kappa'-1} \{ C|p_1| + x|q_1| \}.$$

From this formula the convergence of $(4'')$ when $m = m_1$ follows, and we get the inequalities:

$$\begin{aligned} |Y_{m_1\,1}| &\leqq \{3M(x)\}^{m_1-1} C^{m_1-1} x^{R\kappa'} \int_0^x \{ C|p_1| + x|q_1| \} |\log x| dx \\ &\qquad + \{3M(x)\}^{m_1-1} C^{m_1-1} x^{R\kappa'} |\log x| \int_0^x \{ C|p_1| + x|q_1| \} dx \\ &\leqq 2M(x) \{3M(x)\}^{m_1-1} C^{m_1-1} x^{R\kappa'} \leqq \{3M(x)\}^{m_1} C^{m_1} x^{R\kappa'}, \\ |Y'_{m_1\,1}| &\leqq \{3M(x)\}^{m_1-1} C^{m_1-1} |\kappa'| \, x^{R\kappa'-1} \int_0^x \{ C|p_1| + x|q_1| \} |\log x| \, dx \\ &\qquad + \{3M(x)\}^{m_1-1} C^{m_1-1} x^{R\kappa'-1} (|\kappa'| \, |\log x| + 1) \int_0^x \{ C|p_1| + x|q_1| \} dx \\ &\leqq 3M(x) \{3M(x)\}^{m_1-1} C^{m_1} x^{R\kappa'-1} \leqq \{3M(x)\}^{m_1} C^{m_1+1} x^{R\kappa'-1}, \end{aligned}$$

and these are the inequalities $(5')$ when $m = m_1$.

Let us now define the functions φ_{m1} and $\bar{\varphi}_{m1}$ as follows:

$$Y_{m1} = x^{\kappa'} \varphi_{m1}(x), \qquad Y'_{m1} = x^{\kappa'-1} \bar{\varphi}_{m1}(x).$$

These functions satisfy the inequalities:

$$|\varphi_{m1}| \leqq \{3M(x)\}^m C^m, \qquad |\bar{\varphi}_{m1}| \leqq \{3M(x)\}^m C^{m+1},$$

and therefore approach zero with x when $m > 0$. We will define them as having in this case the value zero, when $x = 0$.

We will now take c so small that $3M(c)C<1$. Then the inequalities last written show that the series:

$$\sum_{m=0}^{m=\infty}\varphi_{m1}(x), \qquad \sum_{m=0}^{m=\infty}\bar{\varphi}_{m1}(x)$$

are absolutely and uniformly convergent throughout the interval $0\leqq x\leqq c$. We then prove, precisely as on pp. 44, 45, that if

$$y_1=\sum_{m=0}^{m=\infty}Y_{m1}, \quad y_1'=\sum_{m=0}^{m=\infty}Y_{m1}',$$

then y_1 is a solution of (1) and y_1' is its derivative.

(*b*) *We start from the function* $Y_{02}=\eta_2$ and compute Y_{12}, Y_{22}, $\cdots$ by means of (3). This gives:

$$Y_{m2}=f_{m2}^{(1)}\eta_1+f_{m2}^{(2)}\eta_2, \qquad Y_{m2}'=f_{m2}^{(1)}\eta_1'+f_{m2}^{(2)}\eta_2',$$

where $f_{m2}^{(1)}$ and $f_{m2}^{(2)}$ are given by the two integrals in (4″).

As in the first part of this section we assume that $1/c\geqq e$. Besides the two quantities C and M there introduced we need here the function:

$$N(x)=\int_0^x\{C\,|p_1|+x\,|q_1|\}(\log x)^2dx.$$

Here again we use the method of mathematical induction to prove that the integrals (4″) converge, and also to establish the inequalities:

$$(5'')\qquad \begin{aligned}|Y_{m2}|&\leqq 6N(x)\{3M(x)\}^{m-1}C^{m-1}x^{Rx'},\\ |Y_{m2}'|&\leqq 6N(x)\{3M(x)\}^{m-1}C^{m}x^{Rx'-1}.\end{aligned}\qquad (m=1,\ 2,\ \ldots)$$

We first show directly that when $m=1$ the integrals (4″) converge and the formulæ (5″) hold. This little piece of computation we will omit for the sake of brevity. Then we show that if when $m\leqq m_1-1$ the integrals (4″) converge and the formulæ (5″) hold, the same will be true when $m=m_1$. The details of this work we also omit as they are almost identical with the work at the corresponding point in the treatment of case (*a*).

We next let:

$$Y_{m2}=x^{\kappa'}\log x\,\varphi_{m2}(x), \qquad Y_{m2}'=x^{\kappa'-1}\log x\,\bar{\varphi}_{m2}(x),$$

and obtain the inequalities:

$$\begin{aligned}|\log x|\,|\varphi_{m2}(x)|&\leqq 6N(x)\{3M(x)\}^{m-1}C^{m-1},\\ |\log x|\,|\bar{\varphi}_{m2}(x)|&\leqq 6N(x)\{3M(x)\}^{m-1}C^{m}.\end{aligned}\qquad (m=1,2,\cdots)$$

Since these inequalities will be merely reinforced by omitting the logarithms, we infer from them at once that when $m > 0$:

$$\lim_{x=0} \varphi_{m\,2}(x) = \lim_{x=0} \bar{\varphi}_{m\,2}(x) = 0\,.$$

We will therefore define $\varphi_{m\,2}(0)$ and $\bar{\varphi}_{m\,2}(0)$ as having the value zero when $m > 0$. We see also that the series

$$\sum_{m=0}^{m=\infty} \varphi_{m\,2}(x), \qquad \sum_{m=0}^{m=\infty} \bar{\varphi}_{m\,2}(x)$$

are absolutely and uniformly convergent throughout the interval $0 \leqq x \leqq c$ provided that

$$3M(c)C < 1.$$

From this point on we can reason as we have done before. Without spending more time on details, which offer nothing new, we will state at once the results to which this section leads:

If $\kappa' = \kappa''$, equation (1) *has two solutions y_1 and y_2 which together with their derivatives may be written:*

$$y_1 = x^{\kappa'} E_1(x), \qquad y_2 = x^{\kappa'} \log x\, E_2(x),$$
$$y_1' = x^{\kappa'-1} H_1(x), \qquad y_2' = x^{\kappa'\ 1} \log x\, H_2(x).$$

Here $E_1(x)$, $E_2(x)$, $H_1(x)$, $H_2(x)$ are continuous throughout the interval $0 \leqq x \leqq c$, and $E_1(0) = E_2(0) = 1$, $H_1(0) = H_2(0) = \kappa'$.

The solution y_2 can be thrown into a different form, which not only exhibits the nature of the function in more detail, but is also closely analogous to the form in which this solution is ordinarily written when the coefficients of (1) are analytic functions. In order to obtain this form we write:

$$y_2 - x^{\kappa'} \log x = x^{\kappa'} \sum_{m=1}^{m=\infty} \log x \cdot \varphi_{m\,2}(x),$$

$$y_1 \log x - x^{\kappa'} \log x = x^{\kappa'} \sum_{m=1}^{m=\infty} \log x \cdot \varphi_{m\,1}(x).$$

The terms of the series on the second sides of these equations are strictly speaking not defined when $x = 0$. We will however regard them as having the value 0 at this point. It is then easily shown that each term is continuous throughout the interval $0 \leqq x \leqq c$ and that both series are uniformly convergent throughout this interval. This follows immediately from the inequalities we

have already obtained in the case of the first series, while for the second series we need merely to note that:

$$\log x \cdot M(x) \leqq N(x).$$

Each of these two series, therefore, represents a function which is continuous when $0 \leqq x \leqq c$ and which vanishes when $x = 0$. The same is true of the difference of these two functions which we will denote by $F(x)$. If then we subtract the two equations last written, we get:

$$y_2 = y_1 \log x + x^{\kappa'} F(x)$$

where $F(x)$ is continuous throughout the interval $0 \leqq x \leqq c$ and $F(0) = 0$.

HARVARD UNIVERSITY,
Cambridge, Mass., Dec. 31, 1899.

THE ELLIPTIC σ-FUNCTIONS CONSIDERED AS A SPECIAL CASE OF THE HYPERELLIPTIC σ-FUNCTIONS*

BY

OSKAR BOLZA

The object of the following paper is two-fold:

1) To give a sketch of the theory of the elliptic σ-functions as they appear in the light of the theory of the hyperelliptic σ-functions.

2) To serve as an introduction for a subsequent paper in which an analogous presentation of the theory of the hyperelliptic σ-functions will be given. Only such methods will therefore be used which are capable of an extension to the hyperelliptic case.

§1. *Canonical systems of associated integrals of the first, second and third kind.*†

We start from the algebraic equation

$$y^2 = R(x), \tag{1}$$

where

$$\begin{aligned} R(x) &= A_0x^4 + 4A_1x^3 + 6A_2x^2 + 4A_3x + A_4 \\ &= A_0(x - a_0)(x - a_1)(x - a_2)(x - a_3) \end{aligned} \tag{2}$$

is a biquadratic whose roots are all distinct.

We construct the Riemann-surface T and the canonical cross cuts A_1, A_2 by which T is transformed into the simply connected surface T', as in the adjoined diagram, in which the full lines are drawn in the upper sheet, the dotted lines in the lower sheet.

We shall say that an integral of the first kind, w, and an integral of the

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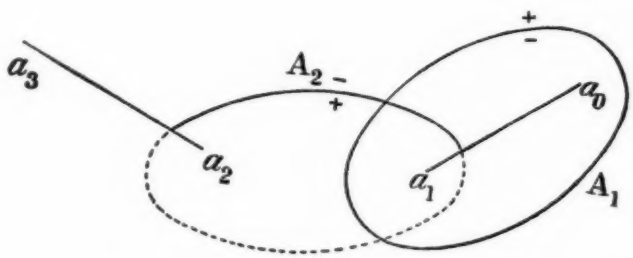
† Compare: BOLZA, *On Weierstrass' Systems of hyperelliptic Integrals of the first and second kind, Chicago Congress Papers*, 1893, p. 1, and the references there given. Also BAKER, *Abel's Theorem*, art. 138–140.

second kind, Z, form a *canonical system*, if the bilinear relation between their periods:

	A_1	A_2
w	$2\omega_1$	$2\omega_2$
Z	$2\eta_1$	$2\eta_2$

has the "canonical" form

$$\eta_1\omega_2 - \eta_2\omega_1 = \frac{\pi i}{2}. \tag{3}$$



There always exists an infinity of such canonical systems, and if w, Z is one such system, every other, $\overline{w}$, $\overline{Z}$, is contained in the form

$$\begin{aligned} \overline{w} &= cw \\ c\overline{Z} &= Z + ew + r(x, y), \end{aligned} \tag{4}$$

where c and e are arbitrary constants and $r(x, y)$ an arbitrary rational function of x and y, (x, y) being the variable upper limit of the integrals.

The most general integral of the third kind $I^{x_1x_0}_{\xi_1\xi_0}$ with the parameters ξ_0, ξ_1 and the limits x_0, x_1 contains one arbitrary constant; its periods at the cross-cuts A_λ are expressible in terms of the integrals of the first and second kind $w^{\xi_1\xi_0}$, $Z^{\xi_1\xi_0}$, taken in J' from ξ_0 to ξ_1. By a proper choice of the constant we can make the integral of the second kind disappear, which determines the constant unambiguously. The *integral of the third kind thus uniquely associated with a given canonical system* we denote by $P_{\xi_1\xi_0}$; its period P_λ at A_λ turns out to be

$$P_\lambda = -2\eta_\lambda w^{\xi_1\xi_0} \qquad (\lambda = 1, 2). \tag{5}$$

As a consequence of (5), this integral is always *commutative:*

$$P^{x_1x_0}_{\xi_1\xi_0} = P^{\xi_1\xi_0}_{x_1x_0}. \tag{6}$$

The *explicit expression* of such a system of associated integrals of the first, second and third kind can be obtained, in the most general way, as follows:

Let $F(x, \xi)$ be an integral function of x and ξ of degree two in each variable, satisfying the following conditions:

$$\begin{aligned} &1) \quad F(\xi, x) = F(x, \xi), \\ &2) \quad F(\xi, \xi) = R(\xi), \\ &3) \quad \left(\frac{\partial F(x, \xi)}{\partial x}\right)_{x=\xi} = \tfrac{1}{2} R'(\xi), \end{aligned} \tag{7}$$

$R'(x)$ being the derivative of $R(x)$; then the integrals

$$\begin{aligned} w^{xx_0} &= \int_{x_0}^{x} \frac{G dx}{y} \\ Z^{xx_0} &= -\frac{1}{G}\int_{x_0}^{x} \frac{y\eta + F(x, \xi)}{2(x-\xi)^2 y} dx + r(x, y) \\ P^{xx_0}_{\xi_1\xi_0} &= \int_{x_0}^{x}\int_{\xi_0}^{\xi_1} \frac{yy' + F(x, x')}{2(x-x')^2 yy'} dx dx', \end{aligned} \tag{8}$$

where G is an arbitrary constant, $r(x, y)$ an arbitrary rational function of x, y, $\eta^2 = R(\xi)$, and $y'^2 = R(x')$, form a system of associated integrals, and every such system can be obtained in this manner.

If we pass from the canonical system w, Z to another $\overline{w}$, $\overline{Z}$ by the transformation (4), the integral $\overline{P}_{\xi_1\xi_0}$ and the function $\overline{F}(x, \xi)$ connected with $\overline{w}$, $\overline{Z}$ are:

$$\begin{aligned} \overline{P}^{xx_0}_{\xi_1\xi_0} &= P^{xx_0}_{\xi_1\xi_0} - e w^{xx_0} w^{\xi_1\xi_0}, \\ \overline{F}(x, \xi) &= F(x, \xi) - 2eG^2(x-\xi)^2. \end{aligned} \tag{9}$$

The following *two special values of* $F(x, \xi)$ are of particular importance for the sequel:

$$F(x, \xi) = A_x^2 A_\xi^2, \tag{10}$$

where symbolically

$$\begin{aligned} R(x) &= A_x^4; \\ F(x, \xi) &= \tfrac{1}{2}[\phi(x)\psi(\xi) + \phi(\xi)\psi(x)], \end{aligned} \tag{11}$$

where

$$R(x) = \phi(x)\psi(x)$$

is a decomposition of $R(x)$ into two quadratic factors.

§2. *Weierstrass' Θ-functions.**

(*a*) *As functions of u.*

With every canonical system of integrals w, Z and every system of cross-cuts A_1, A_2, there is further uniquely associated a function $\Theta\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right](u)$ as follows:

Three constants a, β, τ being defined by

$$2\omega_1 a = \eta_1, \quad 2\beta\omega_1 = 1, \quad \omega_1\tau = \omega_2, \tag{12}$$

$$\Theta\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right](u) = \sum_{\nu=-\infty}^{+\infty} e^{au^2+2\pi i\beta u\left(\nu+\frac{g}{2}\right)+\pi i\tau\left(\nu+\frac{g}{2}\right)^2+\pi ih\left(\nu+\frac{g}{2}\right)}, \tag{13}$$

g and h being integers. The series is convergent since in consequence of our agreements concerning the cross-cuts

$$\Re\left(\frac{\omega_2}{i\omega_1}\right) > 0.$$

The principal properties of these functions are:

$$\begin{aligned} \Theta\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right](u + 2\omega_1) &= (-1)^g e^{2\eta_1(u+\omega_1)}\Theta\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right](u), \\ \Theta\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right](u + 2\omega_2) &= (-1)^h e^{2\eta_2(u+\omega_2)}\Theta\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right](u). \end{aligned} \tag{14}$$

Every integral transcendental function of u which satisfies the two relations (14) differs from $\Theta\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right](u)$ only by a constant factor.

If we pass from one canonical system to another by means of the transformation (4), the Θ-functions corresponding to the two systems are connected by the relation

$$\bar{\Theta}\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right](cu) = e^{\lambda cu^2}\Theta\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right](u). \tag{15}$$

On the other hand, if we pass to another system of cross-cuts by the linear transformation of periods

$$\begin{aligned} \tilde{\omega}_1 &= p\omega_1 + q\omega_2, \\ \tilde{\omega}_2 &= p'\omega_1 + q'\omega_2, \end{aligned} \tag{16}$$

the old and the new Θ-functions are connected by the relation

$$\Theta\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right](u) = C\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right]\cdot\tilde{\Theta}\left[\begin{smallmatrix} \tilde{g} \\ \tilde{h} \end{smallmatrix}\right](u), \tag{17}$$

where $C\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right]$ is independent of u and

$$\begin{aligned} \tilde{g} &= pq + pg + qh, \\ \tilde{h} &= p'q' + p'g + q'h. \end{aligned}$$

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*Compare: WEIERSTRASS, Lectures on elliptic and hyperelliptic functions; SCHOTTKY, *Abel'sche Functionen von drei Variabeln*, §1; BAKER, *Abel's Theorem*, art. 189.

(*b*) *In the Riemann-surface.*

If we substitute in the odd Θ-function for u an integral of the first kind: $u = w^{x\xi}$, the function $\Theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](w^{x\xi})$ vanishes in the point $x = \xi$ and in no other point of the Riemann-surface.

More generally, for every characteristic $\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right]$ there exists a half-period $k\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right]$ such that $\Theta\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right](w^{x\xi} + k\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right])$ vanishes for $x = \xi$ and in no other point of T'. Now $k\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right]$, as every half-period, can be expressed in two different ways in the form

$$k\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right] \equiv \epsilon_0 w^{a_0} + \epsilon_1 w^{a_1} + \epsilon_2 w^{a_2} + \epsilon_3 w^{a_3} \pmod{2\omega_1,\ 2\omega_2}$$

where ϵ_0, ϵ_1, ϵ_2, ϵ_3 are each 0 or 1 and w^x is $w^{x a_0}$.

For the dissection of § 1 the result is

$$\begin{aligned} k\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] &\equiv w^{a_0} + w^{a_1} \equiv w^{a_2} + w^{a_3} \\ k\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] &\equiv w^{a_0} + w^{a_2} \equiv w^{a_3} + w^{a_1} \\ k\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] &\equiv w^{a_0} + w^{a_3} \equiv w^{a_1} + w^{a_2} \end{aligned}$$

With every even characteristic $\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right]$ there is therefore associated a decomposition of the biquadratic R into two quadratic factors: $R = \phi\psi$, called the *algebraic characteristic of the function* $\Theta\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right](u)$.

Setting

$$\phi_\lambda = \text{const.}\ (x - a_0)(x - a_\lambda), \qquad \psi_\lambda = R/\phi_\lambda,$$

we shall denote by $\Theta_{\phi_\lambda\psi_\lambda}(u)$ or shorter $\Theta(u)_\lambda$ the Θ-function with the algebraic characteristic $\phi_\lambda\psi_\lambda$, by k_λ the corresponding constant $k\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right]$, and by $\Theta(u)_0$ or simply $\Theta(u)$ the odd Θ-function, the elements of all the characteristics being reduced to 0 or 1.

If we substitute in (17):

$$u = w^{x\xi} + k\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right],$$

the left-hand side vanishes for $x = \xi$, hence also the right-hand side, therefore

$$\tilde{k}\left[\begin{smallmatrix}\tilde{g}\\\tilde{h}\end{smallmatrix}\right] \equiv k\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right], \tag{18}$$

i. e., the algebraic characteristic remains unaltered under a linear transformation of the periods.

From the properties of the function $\Theta(w^{x\xi} + k_\lambda)_\lambda$ follows the theorem well known in the theory of abelian functions:

$$\frac{\Theta(w^{x\xi} + k_\lambda)_\lambda \Theta(w^{x_0\xi_0} + k_\lambda)_\lambda}{\Theta(w^{x\xi_0} + k_\lambda)_\lambda \Theta(w^{x_0\xi} + k_\lambda)_\lambda} = e^{P^{x x_0}_{\xi \xi_0}}, \tag{19}$$

for $\lambda = 0, 1, 2, 3$ with $k_0 = 0$, $P^{xx_0}_{\xi\xi_0}$ being the integral of the third kind associated with the canonical system w, Z from which the Θ-functions are derived.

Now determine with KLEIN* ξ, ξ_0 by means of the addition-theorem so that

$$\left.\begin{aligned} w^{x\xi_0} + k_\lambda &\equiv 0 \\ w^{x_0\xi} + k_\lambda &\equiv 0 \end{aligned}\right\} \text{ mod } (2\omega_1, 2\omega_2),$$

λ having one of the values 1, 2, 3, and substitute the values of ξ, ξ_0 in (19). The result is *Klein's Theorem*

$$\frac{\Theta^2(w^{xx_0})_\lambda}{\Theta^2(0)\lambda} = \frac{\left(\sqrt{\phi_\lambda(x)}\sqrt{\psi_\lambda(x_0)} + \sqrt{\phi_\lambda(x_0)}\sqrt{\psi_\lambda(x)}\right)^2}{4yy_0} e^{P^{\bar{x}\bar{x}_0}_{xx_0}} \tag{20}$$

where $\bar{x}$, $\bar{x}_0$ are the conjugate places to x, x_0 in T and also the path $\bar{x}_0\bar{x}$ is conjugate to the path x_0x.

The consideration of the zeros and poles of the quotient

$$\frac{\Theta^2(w^{xx_0})}{\Theta^2(w^{xx_0})_\lambda}$$

leads to an analogous expression for the odd function

$$\Theta^2(w^{xx_0}) = \gamma \frac{(x - x_0)^2}{yy_0} e^{P^{\bar{x}\bar{x}_0}_{xx_0}} \tag{21}$$

γ being a constant whose value is found by expansion according to powers of $x - x_0$:

$$\gamma = G^2\Theta'^2(0) \tag{22}$$

$\Theta'(u)$ denoting the derivative of $\Theta(u)$.

§ 3. *The Al-functions and the functions $p(u)_\lambda$.*†

We now *define* in accordance with a notation used by WEIERSTRASS:

$$Al(u)_\lambda = \frac{\Theta(u)_\lambda}{\Theta(0)_\lambda}, \qquad Al(u) = \frac{\Theta(u)}{G\Theta'(0)} \qquad (\lambda = 1, 2, 3) \tag{23}$$

From the properties of the Θ-functions flow at once the following theorems concerning the *Al*-functions:

*Mathematische Annalen, vol. 27, p. 447.

†Compare BOLZA, American Journal of Mathematics, vol. 17, p. 26.

1) *If we pass from one canonical system of integrals to another* by the transformation (4), the Al-functions belonging to the two systems are connected by the relation

$$\overline{Al}(cu)_\lambda = e^{\frac{1}{2}eu^2} Al(u)_\lambda \qquad (\lambda = 0, 1, 2, 3). \tag{24}$$

2) If we pass to another system of cross-cuts *by a linear transformation of the periods, each Al-function remains unchanged.*

3) If we replace u by the integral of the first kind w^{xx_0}, we obtain:

$$Al^2(w^{xx_0})_\lambda = \frac{[\sqrt{\phi_\lambda(x)}\ \sqrt{\psi_\lambda(x_0)} + \sqrt{\phi_\lambda(x_0)}\ \sqrt{\psi_\lambda(x)}]^2}{4yy_0} e^{P^{\bar{x}\bar{x}_0}_{xx_0}} \qquad (\lambda = 1, 2, 3) \tag{25}$$

$$Al^2(w^{xx_0}) = \frac{(x - x_0)^2}{yy_0} e^{P^{\bar{x}\bar{x}_0}_{xx_0}} \tag{26}$$

4) The formulæ for the addition of periods to the argument have exactly the same form as for the Θ-functions.

We further *define* $p(u)_\lambda$ by the equation

$$p(u)_\lambda = -\frac{d^2 \log Al(u)_\lambda}{du^2}, \qquad (\lambda = 0, 1, 2, 3) \tag{27}$$

and write $p(u) = p(u)_0$.

Now take the second logarithmic derivative of (19) with respect to x and ξ, we obtain

$$G^2 p(w^{x\xi}) = \frac{y\eta + F(x, \xi)}{2(x - \xi)^2}. \tag{28}$$

From this fundamental formula follow a number of consequences by giving ξ; or x and ξ special values:

$$p(w^{a_\lambda a_0}) = \frac{F(a_\lambda, a_0)}{2(a_\lambda - a_0)^2};$$

we denote this constant by

$$p(w^{a_\lambda a_0}) = \epsilon_\lambda \qquad (\lambda = 1, 2, 3).$$

Then further

$$G^2[p(w^{xa_0}) - \epsilon_\lambda] = \tfrac{1}{4} \frac{R'(a_0)}{a_0 - a_\lambda} \frac{x - a_\lambda}{x - a_0} = \frac{Al^2(w^{xa_\lambda})_\lambda}{Al^2(w^{xa_0})} \tag{29}$$

and after differentiation

$$G^3 p'(w^{x_0}) = -\tfrac{1}{4}\frac{R'(a_0)y}{(x-a_0)^2}. \tag{30}$$

Hence follow the equations

$$\begin{aligned} G^2[p(u)-\epsilon_\lambda] &= \frac{Al^2(u)_\lambda}{Al^2(u)}, \\ p'^2(u) &= 4\big(p(u)-\epsilon_1\big)\big(p(u)-\epsilon_2\big)\big(p(u)-\epsilon_3\big), \\ &= 4p^3(u)-\gamma_1 p^2(u)-\gamma_2 p(u)-\gamma_3 \qquad \text{say}, \\ G^3 p'(u) &= -\frac{2Al(u)_1\, Al(u)_2\, Al(u)_3}{Al^3(u)}. \end{aligned} \tag{31}$$

We can use the differential equation (31) to determine the first terms in the expansion of $p(u)$ and obtain

$$p(u) = u^{-2} + \tfrac{1}{12}\gamma_1 + \cdots.$$

Hence

$$\begin{aligned} Al(u) &= G^{-1}[u - \tfrac{1}{24}\gamma_1 u^3 + \cdots], \\ Al(u)_1 \cdot Al(u)_2 \cdot Al(u)_3 &= 1 - \tfrac{1}{8}\gamma_1 u^2 + \cdots. \end{aligned} \tag{32}$$

The last expansion can also be obtained independently of the differential equation from the result that

$$p(0)_\lambda = -Al''(0)_\lambda = \epsilon_\lambda;$$

hence

$$Al(u)_\lambda = 1 - \tfrac{1}{2}\epsilon_\lambda u^2 + \cdots. \tag{33}$$

§ 4. *The σ-functions.**

So far no special assumption has been made concerning the canonical system of integrals of the first and second kind with which the Al-functions are associated.

We now propose so to determine the canonical system that in the expansions (31) the second term disappears.

According to (9) and (10) we can always write $F(x, \xi)$ in the form

$$F(x, \xi) = A_x^2 A_\xi^2 - 2eG^2(x-\xi)^2;$$

* Compare KLEIN, *Ueber hyperelliptische Sigmafunctionen*, Mathematische Annalen, vol. 27.

then ϵ_λ takes the form

$$G^2\epsilon_\lambda = \tfrac{1}{4}\frac{R'(a_0)}{a_\lambda - a_0} + \tfrac{1}{24}R''(a_0) - e = -\tfrac{1}{6}(\phi_\lambda, \psi_\lambda)_2 - e$$

and therefore

$$\gamma_1 = 4(\epsilon_1 + \epsilon_2 + \epsilon_3) = 12eG^{-2}.$$

Hence in order to make $\gamma_1 = 0$, we must choose $e = 0$; we further choose $G = 1$ and thus obtain the result:

For the canonical system characterized by

$$w = \int \frac{dx}{y}, \quad F(x, \xi) = A_x^2 A_\xi^2 \tag{34}$$

the following simplifications take place in the expansions of the Al- and p-functions:

1) In the expansion of pu the constant term disappears.

2) In the expansion of $Al(u)$ the first term is u, the term in u^3 disappears.

3) In the expansion of the product $Al(u)_1 \cdot Al(u)_2 \cdot Al(u)_3$ the term in u^2 disappears.

4) At the same time the differential equation for pu simplifies to

$$p'^2(u) = 4p^3(u) - g_2 p(u) - g_3 \tag{35}$$

where g_2, g_3 are the quadratic and cubic invariants of $R(x)$.

The last result shows that the *Al- and p-functions corresponding to this particular canonical system are identical with Weierstrass' σ- and $\wp$-functions.*

But there is still another and even more important consequence of this special choice of the canonical system: If we introduce homogeneous variables, $F(x, \xi) = A_x^2 A_\xi^2$ becomes a covariant of $R(x)$, and *among all possible forms of $F(x, \xi)$ this is the only integral rational covariant of $R(x)$.* Hence follows further that the expressions (26) and (28) for $\sigma(w^{xx_0})$ and $\wp(w^{xx_0})$ are covariants of $R(x)$ with the two sets of variables x, x_0 of index -1 and $+2$ respectively. Substituting therefore *in the expression for σu and $\wp u$, w^{xx_0}* for u, we see that *the coefficients are rational integral invariants of $R(x)$*, and that if we write

$$\sigma u = \sum_{n=0}^{\infty} a_n \frac{u^{2n+1}}{(2n+1)!}, \qquad \wp u = \frac{1}{u^2} + \sum_{n=1}^{\infty} b_n \frac{u^{2n-2}}{(2n-2)!}$$

a_n and b_n are of index $2n$. This reveals the apriori reason why in the expansions of σu and $\wp u$ the second term is missing, viz: because there exists no rational integral invariant of the biquadratic of index 2.

Similar results hold for the functions $\sigma(u)_\lambda$ with respect to the simultaneous system ϕ_λ, ψ_λ.

If we wish to make the second term in the expansion of $Al(u)_\lambda$ equal to zero, we must choose for $F(x, \xi)$ the value

$$F(x, \xi) = \tfrac{1}{2}[\phi_\lambda(x)\psi_\lambda(\xi) + \phi_\lambda(\xi)\psi_\lambda(x)] .$$

The coefficients of the expansion of $Al(u)_\lambda$ will then still be integral rational invariants of the system ϕ_λ, ψ_λ, and the Al-function so determined is connected with $\sigma(u)_\lambda$ by the relation

$$Al(u)_\lambda = e^{\frac{1}{2}e_\lambda u^2}\sigma(u)_\lambda .$$

It is essentially the same as the function $Al(u)$ used by WEIERSTRASS in his first paper on elliptic functions (Werke I, p. 1).

§ 5. *The partial differential equations for the σ-functions.**

If we differentiate an integral of the first or second kind with respect to one of the roots of $R(x)$, a, the result is an integral of the second kind. Hence we may write for any canonical system w, Z:

$$\frac{\partial w}{\partial a} = kw - nZ + r_1 ,$$
$$\frac{\partial Z}{\partial a} = -lw - k'Z + r_2 ,$$

where k, k', l, n are constants, and r_1, r_2 are rational functions of x, y.

Integrating around closed paths independent of a, we obtain for the periods:

$$\frac{\partial \omega_\mu}{\partial a} = k\omega_\mu - n\eta_\mu$$
$$\frac{\partial \eta_\mu}{\partial a} = -l\omega_\mu - k'\eta_\mu \qquad (\mu = 1, 2) \tag{36}$$

and the relation (3) shows that $k' = k$.

From (36) and the definition of the Θ-functions we easily derive the *partial differential equation for* Θ_λ

$$\frac{\partial \Theta_\lambda}{\partial a} = -\tfrac{1}{2}\Theta_\lambda[lu^2 + 2na] - ku\frac{\partial \Theta_\lambda}{\partial u} + \tfrac{1}{2}n\frac{\partial^2 \Theta_\lambda}{\partial u^2} \qquad (\lambda = 0, 1, 2, 3). \tag{37}$$

* Compare: WEIERSTRASS, Werke II, p. 244; BOLZA, American Journal of Mathematics, vol. 21, p. 107, and the references there given; also a continuation of that paper in one of the forthcoming numbers of the American Journal.

From this partial differential equation a number of important consequences can be derived.

A. Concerning the odd σ-function.

For KLEIN's canonical system (34), the constants k, l, n have the following values:

$$k = -\tfrac{1}{12}\frac{R''(a)}{R'(a)} = -\tfrac{1}{12}\frac{\partial}{\partial a}\log\Delta,$$

(38)

$$l = -\frac{g_2}{6R'(a)},\quad n = \frac{2}{R'(a)},$$

Δ being the discriminant of R: $\Delta = g_2^3 - 27g_3^2$.

(*a*) *Value of* $\Theta'(0)$.

Differentiating (37) and putting $u = 0$ we obtain for $\lambda = 0$:

$$\frac{\partial}{\partial a}\log\omega_1^{-\frac{1}{2}}\Theta'(0) = -\tfrac{3}{2}k.$$

Hence

$$\Theta'(0) = C\omega_1^{\frac{1}{2}}\Delta^{\frac{1}{4}} \tag{39}$$

C being independent of the a's. The value of C is found by a limiting process:

$$C = \pi^{-\frac{1}{2}}.$$

(*b*) *Second proof of the invariantive properties of the coefficients in the expansion of σu.*

From (37) we further derive the following partial differential equation for σu:

$$R'(a)\frac{\partial\sigma}{\partial a} = \tfrac{1}{12}g_2u^2\sigma + \tfrac{1}{12}R''(a)\left[u\frac{\partial\sigma}{\partial u} - \sigma\right] + \frac{\partial^2\sigma}{\partial u^2}. \tag{40}$$

Here we substitute for σ its expansion; then we obtain for the quotient,

$$I_n = a_n\Delta^{-\frac{n}{6}},$$

$$\sum_{(a)}\frac{\partial I_n}{\partial a} = 0,\quad \sum_{(a)}a\frac{\partial I_n}{\partial a} = 0,\quad \sum_{(a)}a^2\frac{\partial I_n}{\partial a} = 0, \tag{41}$$

the summations extending over the four roots of $R(x)$. Besides I_n is a homogeneous function of dimension zero of the coefficients of $R(x)$; but these properties characterize I_n as an absolute invariant of $R(x)$; hence a_n *is an invariant of index* $2n$.

(c) *Recursion formula for the expansion of* σu.

Form the differential equation (40) for the four roots of $R(x)$, and introduce the coefficients of $R(x)$ instead of the roots. The final result is: *

If H denotes the Hessian of $R(x)$ and H_i its coefficients, and if the operator δ is defined by

$$\delta = \sum_{i=0}^{4} H_i \frac{\partial}{\partial A_i},$$

σu satisfies the partial differential equation

$$2\delta\sigma = \tfrac{1}{12} g_2 u^2 \sigma + \frac{\partial^2 \sigma}{\partial u^2}, \tag{42}$$

which furnishes for the coefficients the *recursion-formula:*

$$a_{n+1} = 2\delta a_n - \tfrac{1}{6} n(2n+1) g_2 a_{n-1}, \tag{43}$$

with $a_0 = 1$, $a_1 = 0$; for its application notice that

$$\delta g_2 = 6 g_3, \quad \delta g_3 = \tfrac{1}{3} g_2^2.$$

B. Concerning the even σ-functions.

To obtain the analogous results for the even σ-functions, it is simpler to start from the canonical system characterized by (11) and $G = 1$. For this system the values of k, l, n are:

$$k = -\tfrac{1}{4} \frac{\partial \log \Delta_\phi \Delta_\psi}{\partial a}, \quad l = -\tfrac{1}{8} \frac{R_{\phi\psi}}{R'(a)}, \quad n = \frac{2}{R'(a)} \tag{44}$$

$R_{\phi\psi}$ being the resultant of $\phi = \phi_\lambda$ and $\psi = \psi_\lambda$, and Δ_ϕ, Δ_ψ their respective discriminants.

(a) *The Θ_λ-zero-values.*

Put in (37) $u = 0$ and remember that for this canonical system $\Theta'(0)_\lambda = 0$. Then

$$\frac{\partial}{\partial a} \log \omega_1^{-\frac{1}{2}} \Theta(0)_\lambda = -\tfrac{1}{2} k.$$

Hence follows

$$\Theta(0)_\lambda = C_\lambda \omega_1^{\frac{1}{2}} \Delta_\phi^{\frac{1}{4}} \Delta_\psi^{\frac{1}{4}}, \tag{45}$$

where C_λ is independent of the a's and is found to be

$$C_\lambda = 2^{\frac{1}{2}} \pi^{-\frac{1}{2}}.$$

* Already given by PASCAL, Annali di Matematica (2), vol. 17, p. 278.

(*b*) *Second proof for the invariantive properties of the coefficients of the expansion of* $\sigma(u)_\lambda$.

From (37) we derive

$$R'(a)\left[\frac{\partial Al_\lambda}{\partial a} - \tfrac{1}{4}u\frac{\partial Al_\lambda}{\partial u}\frac{\partial \log \Delta_\phi\Delta_\psi}{\partial a}\right] = \tfrac{1}{16}R_{\phi\psi}u^2 Al_\lambda + \frac{\partial^2 Al_\lambda}{\partial u^2}. \tag{46}$$

Hence follows for the coefficients c_n of the expansion

$$Al(u)_\lambda = \sum_{n=0}^{\infty} c_n \frac{u^{2n}}{(2n)!},$$

on setting

$$J_n = c_n \Delta_\phi^{-\frac{n}{2}} \Delta_\psi^{-\frac{n}{2}},$$

$$\sum_{(a)} \frac{\partial J_n}{\partial a} = 0, \quad \sum_{(a)} a\frac{\partial J_n}{\partial a} = 0, \quad \sum_{(a)} a^2\frac{\partial J_n}{\partial a} = 0. \tag{47}$$

Besides J_n is a homogeneous function of dimension zero of the coefficients of ϕ and of the coefficients of ψ. Hence it follows that J_n is an absolute invariant of the system ϕ, ψ; consequently c_n is a simultaneous invariant of ϕ and ψ of index $2n$.

(*c*) *Recursion formula for the expansion of* $Al(u)_\lambda$.

Write (46) for the four roots of $R(x)$, add, and introduce the coefficients of ϕ and ψ instead of their roots. The final result is:

Let D denote the operator

$$D = \tfrac{1}{2}\left[\Delta_\phi \sum_{i=0}^{2} \psi_i \frac{\partial}{\partial \phi_i} + \Delta_\psi \sum_{i=0}^{2} \phi_i \frac{\partial}{\partial \psi_i}\right],$$

and $A_{\phi\psi} = (\phi, \psi)_2$; then $Al(u)_\lambda$ *satisfies the following partial differential equation*

$$D(Al_\lambda) = \tfrac{1}{2}A_{\phi\psi}u\frac{\partial Al_\lambda}{\partial u} + \tfrac{1}{16}R_{\phi\psi}u^2 Al_\lambda + \frac{\partial^2 Al_\lambda}{\partial u^2}. \tag{48}$$

Hence follows for the coefficients c_n in the expansion

$$\sigma(u)_\lambda = e^{-\frac{1}{2}e_\lambda u^2}\sum_{n=0}^{\infty} c_n \frac{u^{2n}}{(2n)!}$$

the *recursion formula*

$$c_{n+1} = D(c_n) - nA_{\phi\psi}c_n - \tfrac{1}{8}n(2n-1)R_{\phi\psi}c_{n-1} \tag{49}$$

with $c_0 = 1$, $c_1 = 0$.

Since

$$D(R_{\phi\psi}) = 0, \quad D(A_{\phi\psi}) = A_{\phi\psi}^2 - R_{\phi\psi} \tag{50}$$

the coefficients are integral functions of $A_{\phi\psi}$ and $R_{\phi\psi}$.

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ON THE GROUPS WHICH ARE THE DIRECT PRODUCTS OF TWO SUBGROUPS*

BY

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If a group is generated by two self-conjugate subgroups which have only identity in common it is said to be the direct product † of these subgroups. It is well known that every operator of one of these groups is commutative with every operator of the other.‡ Evidently one of the necessary conditions that a group (G) may be the direct product of two subgroups is that it contains a subgroup (G_1) such that every operator of G_1 is transformed into its various conjugates under G by the operators of G_1; *i. e.*, each operator of G_1 has in all the same transforms with respect to its own operators as it has with respect to the operators of G. That this condition is not sufficient follows from the cyclical groups whose order is a power of a prime, from the quaternion group, and from many other known groups. This condition is explicitly employed in theoremIII.

THEOREM I.—*If G has a solvable quotient group $G/\mathcal{H}$ such that in the isomorphism of G with $G/\mathcal{H}$ to each operator of $G/\mathcal{H}$ there corresponds one and only one operator of G whose order is a divisor of the order of $G/\mathcal{H}$ then G is the direct product of $\mathcal{H}$ and a subgroup which is simply isomorphic to $G/\mathcal{H}$.*

Let $1, \mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_m \equiv G/\mathcal{H}$ be a series of groups such that each one includes the one which precedes it and some additional operators, each is self-conjugate in $\mathcal{H}_m$, and all the quotient groups $\mathcal{H}_a/\mathcal{H}_{a-1}$ $(a = 1, 2, \cdots, m)$ are abelian.§ Let $1, T_2, T_3, \cdots, T_n$ be all the operators of G whose orders are divisor of the order (h_m) of $G/\mathcal{H}$; $n = h_m$. It is only necessary to prove that these n operators constitute a group.‖ In the isomorphism of G and $G/\mathcal{H}$ each one of these T's evidently corresponds to an operator of the same order

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† As substitution groups these groups have been known for a long time. They occupy a prominent place in the theory of intransitive groups. HÖLDER seems to have been the first to call them (as abstract groups) direct products, Mathematische Annalen, vol. 43, p. 330, 1893.

‡ DYCK, Mathematische Annalen, vol. 22, p. 79, 1883.

§ JORDAN, *Traité des substitutions*, p. 395, 1870.

‖ DYCK, l. c.

in $\mathfrak{G}/\mathfrak{K}$ and all the T's must transform each other in exactly the same manner as the corresponding operators of $\mathfrak{G}/\mathfrak{K}$ transform each other. Since $\mathfrak{K}_1$ is abelian the corresponding T's $(1, T_2, \cdots T_{h_1})$ constitute an abelian group. This group $(\mathfrak{H})$ is a self-conjugate subgroup of $\mathfrak{G}$ since $\mathfrak{K}_1$ is self-conjugate in $\mathfrak{G}/\mathfrak{K}$. This proves the theorem if $m = 1$.

When $m > 1$ we can readily prove the theorem by induction. Suppose that all the T's which correspond to $\mathfrak{K}_a$ $(a < m)$ constitute a self-conjugate subgroup $(\mathfrak{H}^a)$ of $\mathfrak{G}$ and let T_{a+1} be any one of the given n T's that corresponds to an operator of $\mathfrak{K}_{a+1}$ but is not contained in $\mathfrak{H}^a$. T_{a+1} and $\mathfrak{H}^a$ clearly generate a group whose order is a divisor of n. This group $\mathfrak{H}^{a+1}$ must be transformed into itself by all the operators of $\mathfrak{G}$ which correspond to $\mathfrak{K}_{a+1}$ since $\mathfrak{K}_{a+1}/\mathfrak{K}_a$ is abelian. If the order of $\mathfrak{H}^{a+1}$ is less than that of $\mathfrak{K}_{a+1}$ we let T_β be any one of a given n T's which corresponds to an operator of $\mathfrak{K}_{a+1}$ and is not contained in $\mathfrak{H}^{a+1}$, T_β and $\mathfrak{H}^{a+1}$ will clearly generate a larger group whose order is a divisor of n and which is self-conjugate in the subgroup of $\mathfrak{G}$ which corresponds to $\mathfrak{K}_{a+1}$. Hence we observe, by induction, that all the T's which correspond to $\mathfrak{K}_{a+1}$ constitute a self-conjugate subgroup of $\mathfrak{G}$ provided all those which correspond to $\mathfrak{K}_a$ constitute such a self-conjugate subgroup. We proved above that all those which correspond to $\mathfrak{K}_1$ constitute such a self-conjugate subgroup. Hence the proof is complete.

THEOREM II.—*If the order of a group $\mathfrak{K}$ is mn (m and n being prime to each other), and if $\mathfrak{K}$ contains a subgroup $\mathfrak{M}$ of order m which has the property that for every operator K of $\mathfrak{K}$ there is an operator M' of $\mathfrak{M}$ such that for every operator M of $\mathfrak{M}$ the transforms by K and by M' are equal,*

$$(K^{-1}MK = M'^{-1}MM'),$$

and if the quotient group $\mathfrak{K}/\mathfrak{M}$ is solvable, then $\mathfrak{K}$ is the direct product of its subgroups of orders m and n respectively.

It is clear that $\mathfrak{M}$ is a self-conjugate subgroup of $\mathfrak{K}$. Arranging all the operators of $\mathfrak{K}$ in the following manner:

$$\begin{array}{ccccccc} 1 & S_2 & S_3 & S_4 & \cdots & S_m \\ T_2 & S_2T_2 & S_3T_2 & S_4T_2 & \cdots & S_mT_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ T_n & S_2T_n & S_3T_n & S_4T_n & & S_mT_n \end{array}$$

(the first row being composed of the operators of $\mathfrak{M}$) we proceed to prove that there is one and only one operator whose order is prime to m in each row. If $\mathfrak{M}$ contains just k' operators that are commutative with each operator of $\mathfrak{M}$ there must be just k' operators in each one of the given rows that are commuta-

tive with each operation of $\mathfrak{m}$; for if T_a transforms the operators of the first row according to a given substitution there must be k' operators in the first row which transform all the operators of this row according to the inverse of this substitution. These k' operators multiplied into T_a give the required k' operators (of the row which contains T_a) that are commutative with each operator of $\mathfrak{m}$. Suppose T_a is one of these k' operators and that γ is the order of the corresponding operator in the quotient group $\mathfrak{K}/\mathfrak{m}$. From

$$(S_{a_1} T_a)^\gamma = S_{a_1}^\gamma T_a^\gamma$$

and the fact that γ is prime to m, it follows that we obtain all the operators of $\mathfrak{m}$ by raising all the operators of the row which contains T_a to the γ power. This proves that there is one and only one operator whose order is prime to m in each row and that the order of this operator is the same as that of the corresponding operator in $\mathfrak{K}/\mathfrak{m}$. Hence it follows directly from theorem I that $\mathfrak{K}$ is the direct product of $\mathfrak{m}$ and the subgroup of order n.

For use in the proof of theorem III one notices that the second hypothesis of theorem III is fulfilled if the operators M of the group $\mathfrak{m}$ are individually self-conjugate under any certain n extenders of $\mathfrak{m}$ to $\mathfrak{K}$.

THEOREM III.—*If the order of a solvable group $\mathfrak{K}$ is $h = mp^a$ (where p is a prime and a and m are integers, m being prime to p) and if all the conjugates under $\mathfrak{K}$ of every operator A of a subgroup $\mathfrak{A}$ of order p^a are conjugates of A under $\mathfrak{A}$, then $\mathfrak{K}$ is the direct product of $\mathfrak{A}$ and a subgroup $\mathfrak{M}$ of order m, and further $\mathfrak{K}$ contains certain self-conjugate subgroups $\mathfrak{N}_\gamma$ of the orders $n_\gamma = mp^\gamma$ ($\gamma = 0, 1, \cdots, a-1$).*

Each one of the self-conjugate subgroups of $\mathfrak{A}$ is clearly also a self-conjugate subgroup of $\mathfrak{K}$. Hence there must be a p^β, 1 (β being one of the numbers $1, 2, \cdots, a-1$) isomorphism between $\mathfrak{K}$ and each one of a series of groups

$$\mathfrak{G}_{a-1}, \mathfrak{G}_{a-2}, \mathfrak{G}_{a-3}, \cdots, \mathfrak{G}_0$$

whose orders are divisible by $p^{a-1}, p^{a-2}, p^{a-3}, \cdots, p^0$ respectively but by no higher power of p. According to theorem II, $\mathfrak{G}_1$ is the direct product of its subgroups of orders p and m. Suppose that $\mathfrak{G}_\beta$ ($\beta < a-1$) is the direct product of its subgroups of orders p^β and m. To the subgroup of order m in $\mathfrak{G}_\beta$ there must correspond a subgroup of order pm in $\mathfrak{G}_{\beta+1}$. From theorem II it follows that this subgroup of order pm is the direct product of its subgroups of orders p and m. $\mathfrak{G}_{\beta+1}$ must therefore be the direct product of its subgroups of orders $p^{\beta+1}$ and m whenever $\mathfrak{G}_\beta$ is the direct product of its subgroups of orders p^β and m. Since $\mathfrak{G}_1$ is the direct product of its subgroups of orders p and m it

follows from what has just been proved that $\mathfrak{H}$ must be the direct product of its subgroups of orders p^a and m. The remainder of the theorem follows directly from this property.

Every group of order $2m$, m being any odd number, contains a self-conjugate subgroup of order m.* If it contains a self-conjugate subgroup of order 2 it is evidently the direct product of these two self-conjugate subgroups.

If a group is the direct product of two subgroups its group of cogredient isomorphisms is the direct product of the groups of cogredient isomorphisms of these two subgroups.† The converse of this is not generally true but we may readily prove that it is true if the conditions mentioned in the following theorem are satisfied.

THEOREM IV.—*If the group of cogredient isomorphisms* ($\mathfrak{G}'$) *of a group* ($\mathfrak{G}$) *is the direct product of two subgroups* ($\mathfrak{M}$, $\mathfrak{N}$) *whose orders* (m, n) *are prime to each other, then* $\mathfrak{G}$ *is also the direct product of two subgroups.*

COROLLARY.—*If a group* ($\mathfrak{L}$) *has an abelian group of cogredient isomorphisms whose order is not a power of a single prime number then* $\mathfrak{L}$ *is the direct product of two subgroups.*‡

All the operators of $\mathfrak{G}$ which are commutative with each one of its operators constitute the subgroup ($\mathfrak{G}_1$) of order g_1 which corresponds to identity of $\mathfrak{G}'$ in the isomorphism of $\mathfrak{G}$ and $\mathfrak{G}'$. Let

$$m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots, \quad n = q_1^{\beta_1} q_2^{\beta_2} \cdots, \quad g_1 = p_1^{\alpha_1'} p_2^{\alpha_2'} \cdots q_1^{\beta_1'} q_2^{\beta_2'} \cdots r_1^{\gamma_1} r_2^{\gamma_2} \cdots,$$

where

$$p_1, p_2, \cdots, \quad q_1, q_2, \cdots, \quad r_1, r_2, \cdots,$$

are distinct prime numbers; and let G_a be any operator of $\mathfrak{G}$ which corresponds to a given operator M_a of $\mathfrak{M}$ in the given isomorphism of $\mathfrak{G}$ and $\mathfrak{G}'$. $G_a^{m_a}$ (m_a being the order of M_a) must occur in $\mathfrak{G}_1$. If the order of $G_a^{m_a}$ contains a factor $(q_1^{\beta_1''} q_2^{\beta_2''} \cdots r_1^{\gamma_1'} r_2^{\gamma_2'} \cdots)$ which is prime to m, $G_a^{m_a}$ must be the direct product of an operator of $\mathfrak{G}_1$ of order $q_1^{\beta_1''} q_2^{\beta_2''} \cdots r_1^{\gamma_1'} r_2^{\gamma_2'} \cdots$ and an operator of order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots$. In this case $\mathfrak{G}_1$ must contain some operator G_1 such that the order of $(G_1 G_a)^{m_a}$ does not contain any factor that is prime to m.

We may therefore assume that to each operator of $\mathfrak{M}$ in the given isomorphism of $\mathfrak{G}$ and $\mathfrak{G}'$ there corresponds some operator of $\mathfrak{G}$ whose order does not involve any factor that is prime to m. Hence $\mathfrak{G}$ contains a subgroup of order $p_1^{\alpha_1+\alpha_1'} p_2^{\alpha_2+\alpha_2'} p_3^{\alpha_3+\alpha_3'} \cdots$ which includes all the operators of $\mathfrak{G}$ whose orders do not

* FROBENIUS, Berliner Sitzungsberichte, 1895, p. 180.

† Bulletin of the American Mathematical Society, vol. 5, p. 294, 1899.

‡ Cf. BURNSIDE, *Theory of Groups of a Finite Order*, p. 115, 1897.

involve any factor that is prime to m. This subgroup must be self-conjugate. In exactly the same way we may prove that $\mathfrak{G}$ contains a self-conjugate subgroup of order $q_1^{\beta_1+\beta'} q_2^{\beta_2+\beta'_2} q_3^{\beta_3+\beta'_3} \cdots$ which includes all the operators of $\mathfrak{G}$ whose orders do not involve any factor that is prime to n. If each of the r's is equal to unity $\mathfrak{G}$ is the direct product of $\mathfrak{m}$ and $\mathfrak{n}$. In general $\mathfrak{G}$ is evidently the direct product of $\mathfrak{m}$, $\mathfrak{n}$ and the subgroup of order $r_1^{\gamma_1} r_2^{\gamma_2} r_3^{\gamma_3} \cdots$ which is contained in $\mathfrak{G}$, and hence it is also the direct product of one of these three groups and the product of the other two.

If a group $\mathfrak{m}$ contains only identity as a self-conjugate operator, then it is simply isomorphic to its group of cogredient isomorphisms. If moreover $\mathfrak{m}$ is a subgroup of a group $\mathfrak{K}$ with the property of the hypothesis of theorem II, then $\mathfrak{K}$ is the direct product of $\mathfrak{m}$ and a subgroup which is simply isomorphic to $\mathfrak{K}/\mathfrak{m}$. This subgroup is composed of all the operators of $\mathfrak{K}$ which are commutative with every operator of $\mathfrak{m}$. In fact, it is evident that all the operators of a group which are commutative with every operator of any one of its self-conjugate subgroups must always constitute a self-conjugate subgroup. This subgroup may evidently be the entire group. In particular, when a complete group is self-conjugate in any group the latter is the direct product of this complete group and the corresponding quotient group.*

If $\mathfrak{G}$ is the direct product of $\mathfrak{G}_1$, $\mathfrak{G}_2$ then any group contained in $\mathfrak{G}_1$ together with any group contained in $\mathfrak{G}_2$ generates a group which is the direct product of these two generating groups. Suppose $\mathfrak{G}$ may be represented as a primitive substitution group. Then the subgroup which includes all its substitutions that do not involve a given element is maximal and not self-conjugate. From this it follows that this subgroup does not contain any self-conjugate subgroup of $\mathfrak{G}$ besides identity. Hence the given subgroup must be formed by establishing a simple isomorphism between the substitution group which are simply isomorphic to $\mathfrak{G}_1$ and $\mathfrak{G}_2$, *i. e., the necessary and sufficient condition that* $\mathfrak{G}$ *can be represented as a primitive substitution group is that* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *are simply isomorphic simple groups of a composite order.*† When this condition is satisfied $\mathfrak{G}$ can evidently be represented as a primitive group in only one way and the degree of this primitive group is the square root of the order of $\mathfrak{G}$.‡

From the preceding paragraph it follows directly that the degrees of the primitive groups which are the direct products of two groups have a (1, 1) correspondence to the simple groups of composite order; the first group of this kind being of degree 60 and order 3600, the second of degree 168 and order 28224,

* HÖLDER, Mathematische Annalen, vol. 46, p. 325.

† Cf. BURNSIDE, *Theory of Groups*, p. 190, 1897.

‡ Cf. MAILLET, *Thèse de Doctorat*, p. 31, Paris, 1892.

etc. These groups can be represented in a large number of different ways as imprimitive groups.

If G_{f_1} and G_{f_2} are the direct products of α and β groups respectively then G_f may be said to be the direct product of these $\alpha + \beta$ groups and it can be represented as the product of any $\alpha + \beta$ transitive substitution groups (written in distinct elements) which satisfy the condition that they are simply isomorphic to all the given $\alpha + \beta$ groups, each being associated with only one of them. Hence the necessary and sufficient condition that G_f is the direct product of γ-groups is, that it contains γ self-conjugate subgroups $(G_{f_1}, G_{f_2}, \cdots, G_{f_\gamma})$ such that the group generated by any δ of them has only identity in common with any one of the remaining $\gamma - \delta$ and that the order of G_f is the product of the orders of these subgroups.* When $\gamma > 2$ it is clearly impossible to represent G_f as a primitive substitution group. Hence the necessary and sufficient condition that a direct product may be represented as a primitive substitution group is that it contains just two factors and that these are simply isomorphic simple groups of composite order.

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*HÖLDER, Mathematische Annalen, vol. 43, p. 330, 1893.

ON CERTAIN CRINKLY CURVES*

BY

ELIAKIM HASTINGS MOORE

Introduction. – In any field of geometric investigation the curves fall roughly into two classes, constituted respectively of the curves ordinarily investigated and of the other curves; these unusual curves are in positive designation the crinkly curves.

In this paper we are to investigate by interplaying graphic and analytic methods (in I) the continuous surface-filling xy-curves: $x = \varphi(t)$, $y = \psi(t)$: of PEANO and HILBERT and (in II) the continuous tangentless yt-curve: $y = \psi(t)$: connected with PEANO'S curve. We define the various curves K as point-for-point limit-curves for $n = \infty$ of certain curves K_n $(n = 1, 2, 3, \cdots)$; these curves K_n are broken-line curves derivable each from the preceding by processes simple and such that the (nodal) extremities of the various n-links of K_n persist as corresponding points and also nodes of the K_{n+1}; thus, the nodes of K_n are points of K; the set of all these nodes (for all n's) is on K everywhere dense. The curves K_n are continuous and approach their point-for-point limit-curve K uniformly; K is accordingly continuous, a conclusion however which is geometrically evident. From the continuity of K and the presence of the set of nodes the properties of K follow in such a way as to appeal vividly to the geometric imagination. Indeed the yt-curve from the simplicity of its geometric definition and from the intuitive clearness of its properties appears to be fit to replace the classical WEIERSTRASS curve as the standard example of continuous curves having no tangents, since, further, we develop closer knowledge of its progressive- and regressive-tangential properties (II §§ 8, 11).

The basal notions of this paper were communicated to Chicago colleagues in February and March, 1899. – Part II has certain relations of content with the interesting paper by STEINITZ, *Stetigkeit und Differentialquotienten*, Mathematische Annalen, vol. 52, pp. 58–69, May 1899. These relations are indicated in the foot-note of II §7. STEINITZ determines a class of continuous functions having for no argument a derivative; he does not broach the question of progressive and regressive derivatives. – [Jan. 17, 1900. Part II has relations of method, but neither of origin nor of content, with the memoir of

* Presented to the Society, August 25, 1899, at the Columbus meeting. Received for publication December 18, 1899.

BRODÉN: *Beiträge zur Theorie der stetigen Functionen einer reellen Veränderlichen*, Journal für Mathematik, vol. 118, pp. 1–60, 1897. As a generalization of the basal remark of BRODÉN (l. c., p. 1) I notice the theorem that every continuous curve in δ dimensions, ($\delta = 2, 3, \cdots$ or $\aleph_0$) is the point-for-point limit-curve for $n = \infty$ of a sequence of inscribed broken-line curves K_n ($n = 1, 2, \cdots$), the sequence of nodes of every K_n corresponding to a sequence of increasing values of the parameter-of-continuity of K, the nodes of every K_n being nodes of K_{n+1}, and the complete set of nodal arguments being everywhere dense on the set of all arguments of points of K.]

I

Continuous surface-filling curves.

1. Defining (as usual) as a real continuous plane curve the locus of points (x, y) whose coördinates are single-valued real continuous functions

$$x = \varphi(t), \quad y = \psi(t)$$

of the real variable t, PEANO,* 1890, by arithmetic † process determined two functions of this kind for the range $0 \leqq t \leqq 1$ such that the corresponding continuous curve C fills the square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$. We give below (§ 7) a geometric determination of Peano's curve.

2. This interesting phenomenon of continuous surface-filling curves HILBERT ‡ in 1891 made luminous to the geometric imagination in the following way:

For every positive integer n the line $0 \leqq t \leqq 1$ is divided into 4^n intervals I_n of length 4^{-n} and the square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$ is divided into 4^n squares S_n of length 2^{-n}. A 1–1 correspondence is effected between the 4^n intervals I_n and the 4^n squares S_n ($n = 1, 2, 3, \cdots$) satisfying the two conditions: (1) to two adjacent intervals I_n correspond two adjacent squares S_n; (2) to the four intervals I_{n+1} of an interval I_n correspond the four squares S_{n+1} of the corresponding square S_n. Then to an infinite sequence $\{I_n\}$ of intervals I_n ($n = 1, 2, 3, \cdots$) in which every interval I_n includes the succeeding interval I_{n+1} corresponds such a sequence $\{S_n\}$ of squares S_n ($n = 1, 2, 3, \cdots$). In accordance with the geometric axioms of the continuity of the line and the plane (or, if

* PEANO, *Sur une courbe, qui remplit toute une aire plane:* Mathematische Annalen, vol. 36, pp. 157–160, 1890.

† The analytic formulas for the Peano functions given by CESÀRO (*Sur la représentation analytiques des régions, et des courbes qui les remplissent:* (Bulletin des Sciences mathématiques, 2d ser., vol. 21, pp. 257–266, 1897) involve the arithmetic function $[u] = E(u)$. (Cesàro's formulas are in error for $t = 1$.)

‡ HILBERT, *Ueber die stetige Abbildung einer Linie auf ein Flächenstück:* Mathematische Annalen, vol. 38, pp. 459–460, 1891.

our geometric phrasings have solely analytic meaning, by a fundamental theorem concerning the system of real numbers), the sequence $\{I_n\}$ determines a point $T\{I_n\}$ of the line lying on its every interval I_n, and the sequence $\{S_n\}$ a point $Z\{S_n\}$ of the square lying on its every square S_n. Such points T and Z are set in correspondence.

This correspondence indeed determines for every point T of the line a definite point Z of the square, and in such a way that as T describes the line Z describes a continuous curve filling the square. – Proof: (α) A point T of the line which is the extremity of no interval determines uniquely a sequence $\{I_n\}$ to which it belongs and hence a point Z belonging to the corresponding sequence $\{S_n\}$ The same is true for the points $T = (0)$, (1). A point T which is common to two adjacent intervals I_n', I_n'' for some definite value of n, $n = \nu$, is common to two adjacent intervals I_n', I_n'' for every $n \geqq \nu$, and belongs to two sequences $\{I_n'\}$, $\{I_n''\}$; the corresponding points $Z\{S_n'\}$, $Z\{S_n''\}$ are, however, identical, since the squares S_n', S_n'' are adjacent for every $n \geqq \nu$. Thus to every T corresponds one Z. (β) Similarly, every Z of the square belongs to one, two, or four sequences $\{S_n\}$ and corresponds to one, two, three, or four points T of the line. (γ) This dependence of $Z = (x, y)$ upon $T = (t)$ determines x and y as single valued functions

$$x = \varphi(t), \quad y = \psi(t) \tag{1}$$

of t for the range $0 \leqq t \leqq 1$. These functions are continuous, since (by condition (1)) for every n

$$|\varphi(t_1) - \varphi(t_2)| \leqq 2 \cdot 2^{-n}, \quad |\psi(t_1) - \psi(t_2)| \leqq 2 \cdot 2^{-n} \tag{2}$$

for every pair of arguments t_1, t_2 of the range $0 \leqq t \leqq 1$ such that

$$|t_1 - t_2| \leqq 4^{-n}. \tag{3}$$

The relation (2) indeed shows that $\varphi(t)$, $\psi(t)$ are uniformly continuous functions of t for the range $0 \leqq t \leqq 1$.

3. One finds that the fundamental correspondence subject to the conditions (1) (2) between the I_n, S_n ($n = 1, 2, 3, \cdots$) involves 8 elements of indetermination for each successive n introduced, – or 2 elements (after the initial 8), if in condition (1) the initial and final intervals I_n of the line are counted as adjacent intervals – and that it becomes uniquely determinate by the stipulation of the sequences of squares corresponding to the sequences of initial and final intervals. In particular, HILBERT gives the correspondence determined by the stipulations

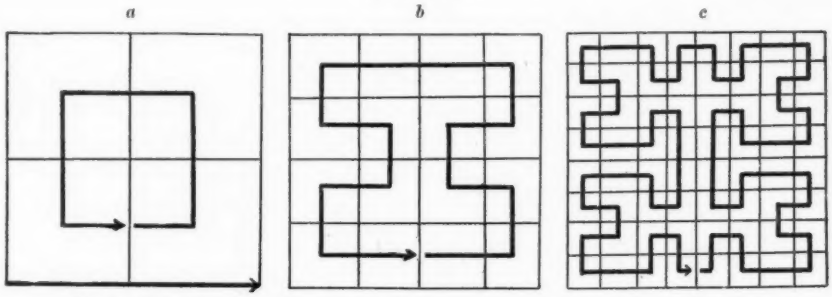
$$\big(\varphi(0),\ \psi(0)\big) = (0,\ 0),\ \big(\varphi(1),\ \psi(1)\big) = (1,\ 0), \tag{4}$$

while the figure 1 below is for one of the two determined by the stipulations

$$\varphi(0) = \varphi(1) = \tfrac{1}{2}\,, \qquad \psi(0) = \psi(1) = 0\,. \tag{5}$$

4. For every particular n to indicate the sequence of the 4^n squares S^n corresponding to the natural sequence of the 4^n intervals I_n, HILBERT draws a broken line passing from the center of the initial square S_n through the centers of the squares in sequence to the center of the final square S_n. – *Let us see how to use this broken line to obtain a still more vivid geometric picture of these continuous surface-filling curves.* We prolong the line across the initial and the

FIGURE 1



final squares. This (completed) broken line C_n having in every square S_n a segment of length 2^{-n} has the length 2^n. We regard the original line C_0 as say the x-side of the original square S_0 and the lines C_n $(n = 0\,,\ 1\,,\ 2\,,\ \cdots)$ as derived each from the preceding by *uniform* stretching (doubling) and suitable "breaking" and locating. Then every point

$$T = T_0 = (t) = (t\,,\ 0) = (x_0\,,\ y_0)$$

of C_0 determines a definite point

$$T_n = (x_n\,,\ y_n) = \Big(\varphi_n(t)\,,\ \psi_n(t)\Big)$$

of C_n $(n = 0\,,\ 1\,,\ 2\,,\ 3\,,\ \cdots)$, and the sequence $\{T_n\}$ of points T_n $(n = 0\,,\ 1\,,\ 2\,,\ 3\,,\ \cdots)$ has as limit the point Z of the surface-filling curve C,

$$Z = (x\,,\ y) = \Big(\varphi(t),\ \psi(t)\Big)\,,$$

$$\varphi(t) = \lim_{n=\infty} \varphi_n(t), \quad \psi(t) = \lim_{n=\infty} \psi_n(t)\,. \tag{6}$$

THEOREM.—*The continuous surface-filling curve C is the point-for-point limit-curve for $n = \infty$ of the broken-line continuous curves $C_n (n = 1, 2, 3, \cdots)$ derived individually from the original line by uniform stretching and suitable breaking and locating.*

I notice further that

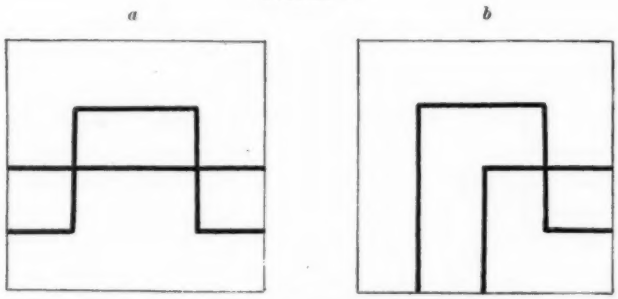
$$(7) \qquad |\varphi_{n_1}(t_1) - \varphi_{n_2}(t_2)| \leqq 2 \cdot 2^{-n}, \quad |\psi_{n_1}(t_1) - \psi_{n_2}(t_2)| \leqq 2 \cdot 2^{-n}$$

for all integers n_1, n_2, $n\ (\geqq 0)$ and numbers t_1, t_2 $(\geqq 0, \leqq 1)$ such that

$$n_1 \geqq n, \quad n_2 \geqq n, \quad |t_1 - t_2| \leqq 4^{-n}.$$

5. The squares S_n are crossed* by the lines C_n, C_{n+1} in essentially only two ways.

FIGURE 2



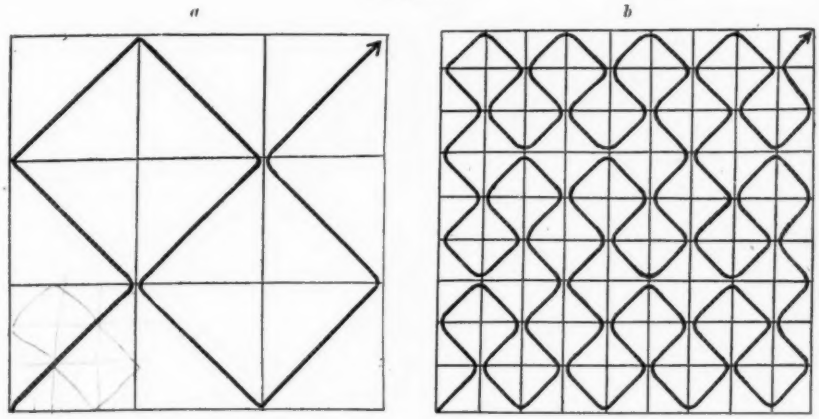
The lines C_n, C_{n+1} are seen to intersect at and only at corresponding points $T_n(t) = T_{n+1}(t)$, which, furthermore, are distinct from their respective corresponding points $T_{n+2}(t)$. Thus on no C_n is there one point $T_n(t)$ in its limit-position $Z(t)$.

6. No simple modification of these approximation curves C_n which preserves their essential relations to the correspondence between the I_n and the S_n and their derivation each from the preceding by uniform stretching and which moreover introduces on every C_n in its every S_n at least one point $T_n(t)$ in its limit-position suggests itself. However without essential change we may replace HILBERT'S basal integer 2 by any *odd* integer $\omega\ (> 1)$ and then find a correspondence between the ω^{2n} intervals I_n and the ω^{2n} squares S_n $(n = 1, 2, 3, \cdots)$ such that the broken lines C_n (may be chosen to) cross the squares S_n on diagonals and that then the extremities of these diagonals are points T_n in final position. The curves $C^{(\omega)}$ so obtained are exactly PEANO'S surface-filling curves.

* For the particular case of $\omega = 3$, the lines C_n are closed and enclose surfaces of area $2^{-1} - 4^{-n}$ $(n = 1, 2, 3, \cdots)$, so that the limit for $n = \infty$ of the area of the surface enclosed by C_n is 2^{-1}, while the point-for-point closed limit-curve C fills the whole unit-square.— The lines C_n are symmetrical with respect to the lines $2x - 1 = 0$, $2y - 1 = 0$.

7. *Geometric determination of Peano's curve* $C^{(\omega)}$ (with figures for the $C^{(3)}$). ω is an odd integer > 1. We stretch the original line $0 \leqq t \leqq 1$ uniformly in the ratio $\sqrt{2} : 1$ and locate it as $C_0^{(\omega)}$ or say C_0 in the diagonal joining the vertices $(x, y) = (0, 0), (1, 1)$ of the initial square S_0. Without changing the extremities of C_0 we uniformly stretch it in the ratio $\omega : 1$ and break and locate it as C_1 (figure 3) traversing on their diagonals the ω^2 squares S_1 of S_0 in such

FIGURE 3



Line $C_1^{(3)}$ Line $C_2^{(3)}$

wise that the S_1 of every vertical column are traversed sequentially. Then a similar treatment of these diagonals of the ω^2 S_1 changes C_1 into C_2. And so on. –

It is convenient to term the diagonals of the C_n *links* (or n-links) and their extremities *nodes* (or n-nodes). The line C_n lies completely on the next line C_{n+1}, but the only points $T_n(t) = T_{n+1}(t)$ are the nodes and the middle points of the links of the C_n.

8. In notations analogous to those of §§ 2, 4 one has first the general relations:

$$(1) \qquad |\varphi(t_1) - \varphi(t_2)| \leqq 2 \cdot \omega^{-n}, \qquad |\psi(t_1) - \psi(t_2)| \leqq 2 \cdot \omega^{-n},$$

$$(2) \qquad |\varphi(t_1) - \varphi_{n_2}(t_2)| \leqq 2 \cdot \omega^{-n}, \qquad |\psi(t_1) - \psi_{n_2}(t_2)| \leqq 2 \cdot \omega^{-n},$$

$$(3) \qquad |\varphi_{n_1}(t_1) - \varphi_{n_2}(t_2)| \leqq 2 \cdot \omega^{-n}, \qquad |\psi_{n_1}(t_1) - \psi_{n_2}(t_2)| \leqq 2 \cdot \omega^{-n}$$

for all integers n_1, n_2, n $(\geqq 0)$ and numbers t_1, t_2 $(\geqq 0, \leqq 1)$ such that

$$n_1 \geqq n, \qquad n_2 \geqq n, \qquad |t_1 - t_2| \leqq \omega^{-2n};$$

$$\varphi(t) = \underset{n=\infty}{\mathbf{L}}\, \varphi_n(t), \qquad \psi(t) = \underset{n=\infty}{\mathbf{L}}\, \psi_n(t) \qquad (0 \leqq t \leqq 1). \tag{4}$$

The uniform continuity of $\varphi(t)$, $\psi(t)$ for $0 \leqq t \leqq 1$ is implied by (1).

One has next the relations connected with $t = \varphi(t) = \psi(t) = 0$:

$$\omega \cdot \varphi_{n+1}(t) = \varphi_n(\omega^2 t), \qquad \omega \cdot \psi_{n+1}(t) = \psi_n(\omega^2 t) \qquad (0 \leqq t \leqq \omega^{-3}), \tag{5}$$

$$\omega \cdot \varphi(t) = \varphi(\omega^2 t), \qquad \omega \cdot \psi(t) = \psi(\omega^2 t) \qquad (0 \leqq t \leqq \omega^{-3}), \tag{6}$$

$$\omega \cdot \varphi_n(t) = \psi_n(\omega t), \qquad \psi_{n+1}(t) = \varphi_n(\omega t) \qquad (0 \leqq t \leqq \omega^{-1}), \tag{7}$$

$$\omega \cdot \varphi(t) = \psi(\omega t), \qquad \psi(t) = \varphi(\omega t) \qquad (0 \leqq t \leqq \omega^{-1}), \tag{8}$$

where n is any integer $\geqq 0$. The relations (7, 8) are derivable from the formulas of §§ 9, 10 and also from the geometrical considerations of II § 5.

One has further the relations connected with $t = \varphi(t) = \psi(t) = \frac{1}{2}$:

$$\varphi_n(\tfrac{1}{2} + u) + \varphi_n(\tfrac{1}{2} - u) = 1, \quad \psi_n(\tfrac{1}{2} + u) + \psi_n(\tfrac{1}{2} - u) = 1 \qquad (|u| \leqq \tfrac{1}{2}), \tag{9}$$

$$\varphi(\tfrac{1}{2} + u) + \varphi(\tfrac{1}{2} - u) = 1, \quad \psi(\tfrac{1}{2} + u) + \psi(\tfrac{1}{2} - u) = 1 \qquad (|u| \leqq \tfrac{1}{2}), \tag{10}$$

$$\begin{aligned} \omega\{\varphi_{n+1}(\tfrac{1}{2} + u) - \tfrac{1}{2}\} &= (-1)^{\frac{\omega-1}{2}}\{\varphi_n(\tfrac{1}{2} + \omega^2 u) - \tfrac{1}{2}\} \\ \omega\{\psi_{n+1}(\tfrac{1}{2} + u) - \tfrac{1}{2}\} &= (-1)^{\frac{\omega-1}{2}}\{\psi_n(\tfrac{1}{2} + \omega^2 u) - \tfrac{1}{2}\} \end{aligned} \qquad (|u| \leqq \tfrac{1}{2}\omega^{-3}), \tag{11}$$

$$\begin{aligned} \omega\{\varphi(\tfrac{1}{2} + u) - \tfrac{1}{2}\} &= (-1)^{\frac{\omega-1}{2}}\{\varphi(\tfrac{1}{2} + \omega^2 u) - \tfrac{1}{2}\} \\ \omega\{\psi(\tfrac{1}{2} + u) - \tfrac{1}{2}\} &= (-1)^{\frac{\omega-1}{2}}\{\psi(\tfrac{1}{2} + \omega^2 u) - \tfrac{1}{2}\} \end{aligned} \qquad (|u| \leqq \tfrac{1}{2}\omega^{-3}), \tag{12}$$

where n is any integer $\geqq 0$. These geometrically evident relations are easily derivable from the formulas of §§ 9, 10.

9. The C_n has $\omega^{2n} + 1$ n-nodes $(x_{n\,k}, y_{n\,k})$ with the n-nodal arguments $t = t_{n\,k}$, where

$$t_{n\,k} = k\omega^{-2n}, \qquad x_{n\,k} = \varphi_n(t_{n\,k}), \qquad y_{n\,k} = \psi_n(t_{n\,k}) \tag{13}$$
$$(k = 0, 1, 2, \cdots, \omega^{2n}).$$

We have

$$\varphi_n(t) = \varphi(t), \qquad \psi_n(t) = \psi(t) \tag{14}$$

for n-nodal arguments t, and further

$$x_{n\,k+1} - x_{n\,k} = \varepsilon_{n\,k}\,\omega^{-n}, \quad y_{n\,k+1} - y_{n\,k} = \eta_{n\,k}\,\omega^{-n} \tag{15}$$
$$(k = 0,\ 1,\ \cdots,\ \omega^{2n}-1),$$

where the $\varepsilon_{n\,k}$, $\eta_{n\,k}$ are signs to be determined. We write k in the scale of ω:

$$k = \{a_1 a_2 a_3 \cdots a_{2n-1}\, a_{2n}\} = \sum_{i=1}^{i=2n} a_i \omega^{2n-i}, \tag{16}$$

where the a_i are integers $0 \leqq a_i < \omega$. For $n = 1$ we have at once

$$\varepsilon_{1\,k} = (-1)^{a_2}, \quad \eta_{1\,k} = (-1)^{a_1}, \tag{17}$$

and so, for the general n, in view of the sequential derivation of the C_{m+1} from the C_m $(m = 1,\ 2,\ \cdots,\ n-1)$,

$$\varepsilon_{n\,k} = (-1)^{\sum_{i=1}^{i=n} a_{2i}}, \quad \eta_{n\,k} = (-1)^{\sum_{i=1}^{i=n} a_{2i-1}}. \tag{18}$$

Setting for any t $(0 \leqq t \leqq 1)$

$$t = t_{n\,k} + \tau_n = k\omega^{-2n} + \tau_n \tag{19}$$

where $0 \leqq \tau_n \leqq \omega^{-2n}$, we have

$$\varphi(t) = \varphi(t_{n\,k}) + \varepsilon_{n\,k}\,\varphi(\tau_n), \quad \psi(t) = \psi(t_{n\,k}) + \eta_{n\,k}\,\psi(\tau_n), \tag{20}$$

$$\varphi_n(t) = \varphi(t_{n\,k}) + \varepsilon_{n\,k}\,\tau_n\omega^n, \quad \psi_n(t) = \psi(t_{n\,k}) + \eta_{n\,k}\,\tau_n\omega^n. \tag{21}$$

10. *Arithmetic-analytic determination of Peano's curve* $C^{(\omega)}$. – For $t = 1$

$$\varphi(1) = 1, \quad \psi(1) = 1. \tag{22}$$

The arguments t, $0 \leqq t < 1$, are written uniquely in the scale of ω,

$$t = \{.\, a_1 a_2 \cdots a_{2n-1} a_{2n} \cdots\} = \sum_{i=1}^{i=\infty} a_i \omega^{-i}, \tag{23}$$

where the a_i are integers $0 \leqq a_i < \omega$ and where in no case is ultimately every a_i $\omega - 1$. For the 1-nodes with $t = \{.\, a_1 a_2 \bar{0}\} = \{.\, a_1 a_2\}$ one has directly

$$\varphi(\{.\, a_1 a_2\}) = x_{1\,k} = x_{1\,\{a_1 a_2\}} = \left(a_1 + \tfrac{1}{2}\{1-(-1)^{a_2}\}\right)\omega^{-1}, \tag{24}$$

$$\psi(\{.\, a_1 a_2\}) = y_{1\,k} = y_{1\,\{a_1 a_2\}} = \tfrac{1}{2}\{1-(-1)^{a_1}\} + (-1)^{a_1} a_2 \omega^{-1}, \tag{25}$$

formulas in accord with § 9 (15, 16, 18). Setting

$$(26)\qquad \varepsilon_0(t)=\gamma_0(t)=1\,,\quad \varepsilon_n(t)=(-1)^{\sum_{i=1}^{i=n}a_{2i}}\,,\quad \gamma_n(t)=(-1)^{\sum_{i=1}^{i=n}a_{2i-1}}$$

$$(n=1,\ 2,\ 3,\ \cdots)$$

and using several formulas of §§ 8, 9, 10, one has

$$(27)\qquad \varphi(t)=\sum_{\nu=0}^{\nu=\infty}\varphi(\{\,.\,a_{2\nu+1}a_{2\nu+2}\})\cdot\varepsilon_\nu(t)\cdot\omega^{-\nu}\,,$$

$$(0\leqq t<1)$$

$$(28)\qquad \psi(t)=\sum_{\nu=0}^{\nu=\infty}\psi(\{\,.\,a_{2\nu+1}a_{2\nu+2}\})\cdot\gamma_\nu(t)\cdot\omega^{-\nu}\,.$$

11. *A continuous representation of the t-ray* $(t\geqq 0)$ *upon the xy-quadrant* $(x\geqq 0\,,\ y\geqq 0)$. – For a simple extension of Peano's curve $C^{(\omega)}$ over the first xy-quadrant corresponding to the extension of the interval $t=0\cdots 1$ over the positive t-ray, wherein we take the relations (6) of § 8 as permanent, we have as definitions for $\varphi(t)$, $\psi(t)$ $(t>1)$

$$(29)\qquad \varphi(t)=\omega^e\varphi(\omega^{-2e}t)\,,\qquad \psi(t)=\omega^e\psi(\omega^{-2e}t)$$

where e is an (any) integer such that $\omega^{2e}\geqq t$. We thus have the desired single-valued continuous functions $\varphi(t)$, $\psi(t)$ for $t\geqq 0$. – The functions $\varphi_n(t)$, $\psi_n(t)$ are likewise extended. The relations (1–4, 5–8) of § 8 and (29) of § 11 hold permanently for all t, t_1, t_2, $e\geqq 0$. – One readily follows these extensions graphically. For instance, to the interval $0\leqq t\leqq\omega^{2e}$ corresponds the square $0\leqq x\leqq\omega^e$, $0\leqq y\leqq\omega^e$.

12. *A continuous representation of the complete t-line upon the complete xy-plane.* – Taking the relations (12) of § 8 as permanent we have as definitions for $\varphi(\frac{1}{2}+u)$, $\psi(\frac{1}{2}+u)$ $(|u|>\frac{1}{2})$

$$(30)\qquad \begin{aligned}\varphi(\tfrac{1}{2}+u)-\tfrac{1}{2}&=(-1)^{\frac{(\omega-1)e}{2}}\omega^e\{\varphi(\tfrac{1}{2}+\omega^{-2e}u)-\tfrac{1}{2}\}\,,\\ \psi(\tfrac{1}{2}+u)-\tfrac{1}{2}&=(-1)^{\frac{(\omega-1)e}{2}}\omega^e\{\psi(\tfrac{1}{2}+\omega^{-2e}u)-\tfrac{1}{2}\}\end{aligned}$$

where e is an (any) integer such that $\omega^{2e}\geqq|u|$. We thus have the desired single-valued continuous functions $\varphi(t)$, $\psi(t)$ for all real values of the argument t. To the interval of length ω^{2e} and center $t=\frac{1}{2}$ corresponds the xy-square of side ω^e and center $(x,\ y)=(\frac{1}{2}\,,\ \frac{1}{2})$. – The functions $\varphi_n(t)$, $\psi_n(t)$ are likewise extended. The relations (1–4, 9–12) of § 8 and (30) of § 12 hold permanently for all t, t_1, t_2, e.

II

Continuous curves having no tangents.

1. The surface-filling xy-curves C

$$x = \varphi(t), \quad y = \psi(t)$$

of PEANO and of HILBERT give rise (as stated by each of them) to continuous xt-, yt-curves X, Y

$$x = \varphi(t), \quad y = \psi(t)$$

having no tangents. – We shall consider the curves $X^{(\omega)}$, $Y^{(\omega)}$ associated with Peano's curve $C^{(\omega)}$ (I §§ 7–10), on the basis however of direct geometric definitions.

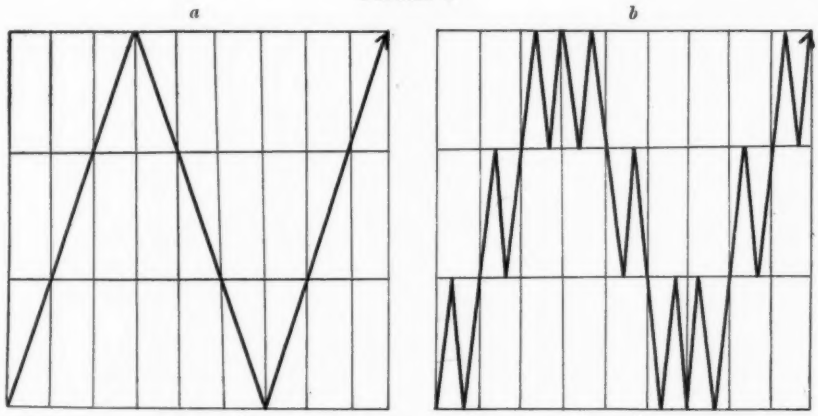
2. We have at once $Y^{(\omega)}$ defined as the point-for-point limit curve for $n = \infty$ of the sequence of (broken-line) curves $Y_n^{(\omega)}$ $(n = 0, 1, 2, \cdots)$

$$y = \psi_n(t)$$

connected with the respective curves $C_n^{(\omega)}$.

3. *Geometric determination of the curves* $Y^{(\omega)}$ (with figures for the $Y^{(3)}$): ω is an odd integer > 1. In the ty-plane we have the fundamental rectangle R_0 $(0 \leqq t \leqq 1, \ 0 \leqq y \leqq 1)$ of dimensions 1×1. We subdivide this into ω^3 rectangles R_1 of dimensions $\omega^{-2} \times \omega^{-1}$, and further into ω^6 rectangles R_2 of

FIGURE 4



Line $Y_1^{(3)}$ Line $Y_2^{(3)}$

dimensions $\omega^{-4} \times \omega^{-2}$, and in general into ω^{3n} rectangles R_n of dimensions $\omega^{-2n} \times \omega^{-n}$ $(n = 0, 1, 2, \cdots)$. A diagonal of a R_n makes with the t-axis an angle whose tangent, the *slope* of the diagonal, is $+\omega^n$ or $-\omega^n$.

The unit-segment $0 \leqq t \leqq 1$ of the t-axis is uniformly stretched and located with the designation Y_0 as the diagonal $(0, 0)(1, 1)$ of R_0. Without change of extremities Y_0 is uniformly stretched, broken, and located as $Y_1^{(\omega)}$ or Y_1 (figure 4) traversing on their diagonals certain ω^2 rectangles R_1; these ω^2 R_1 are one from every vertical column of ωR_1, and such that Y_1 preserves its direction through every node not on the boundaries $y = 0$, $y = 1$ of R_0. A similar treatment of the ω^2 links of the broken line Y_1, diagonals of the ω^2 rectangles R_1, yields the broken line Y_2 composed of ω^4 links diagonals of certain ω^4 R_2. And so on.

The point $(t, y) = (t_0, 0)$ of the t-axis by this process determines for every integer n a point $V_n = (t, y) = \big(t_0, \psi_n(t_0)\big)$. One has

$$|\psi_{n_1}(t_1) - \psi_{n_2}(t_2)| \leqq 2 \cdot \omega^{-n} \tag{1}$$

for all integers n_1, n_2, n $(\geqq 0)$ and numbers t_1, t_2 $(\geqq 0, \leqq 1)$ such that

$$n_1 \geqq n, \qquad n_2 \geqq n, \qquad |t_1 - t_2| \leqq \omega^{-2n}.$$

The sequence $\{V_n\}$ of points V_n $(n = 0, 1, 2, \cdots)$ has for $n = \infty$ the limit-point $V = \big(t, \psi(t)\big)$,

$$V = \mathop{\mathrm{L}}_{n=\infty} V_n, \qquad \psi(t) = \mathop{\mathrm{L}}_{n=\infty} \psi_n(t). \tag{2}$$

One verifies either geometrically or analytically the (notationally implied) relations of the curves and the functions * here introduced with those of part I. In particular, the point $V_n(t)$ of Y_n lies on an n-link of slope $\eta_n(t) \cdot \omega^n$, where, if $t = \{\cdot a_1 a_2 \cdots a_i \cdots\}$ (I § 10),

$$\eta_n(t) = (-1)^{\sum_{i=1}^{i=n} a_{2i-1}}. \tag{3}$$

Only the node-points, $t = \{\cdot a_1 \cdots a_{2n} \dot{0}\} = \{\cdot a_1 \cdots a_{2n}\}$, lie on two links; of these two links the one with slope $\eta_n(t) \cdot \omega^n$ is the right link; the left link has the slope $\pm \eta_n(t) \cdot \omega^n$ according as $a_{2n} \geqq 0$.

4. *Geometric determination of the curve* $X^{(\omega)}$ (with figures for the $X^{(3)}$).— This determination is sufficiently indicated by the graphs of figure 5, the line

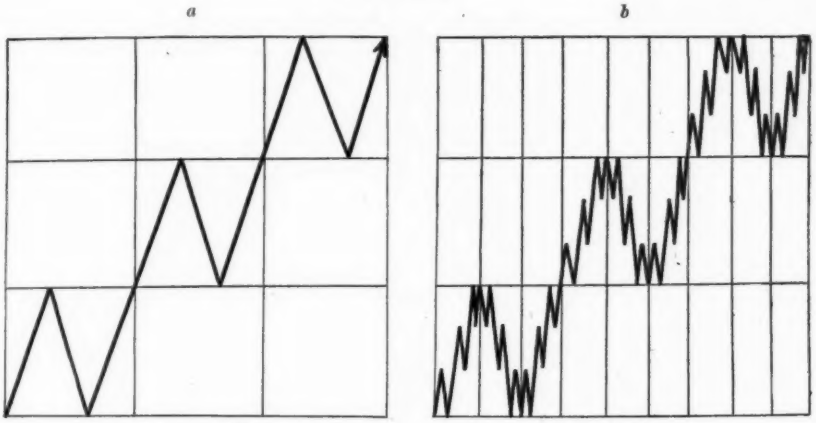
* One notes the formula

$$\psi_1(t) = (-1)^{E(\omega t)}\big(\omega t - E(\omega t)\big) + \tfrac{1}{2}\big(1 - (-1)^{E(\omega t)}\big) \qquad (0 \leqq t \leqq 1),$$

where for real arguments u $E(u)$ denotes the largest integer not greater than u.

$X_0^{(3)}$ being the diagonal of the fundamental square. The functions $\varphi_n(t)$, $\varphi(t)$ $(n = 0, 1, 2, \cdots)$ are introduced with obvious meanings.

FIGURE 5



5. We have

$$\omega \cdot \varphi_{n+1}(t) = \varphi_n(\omega^2 t), \quad \omega \cdot \varphi(t) = \varphi(\omega^2 t) \qquad (0 \leqq t \leqq \omega^{-2}), \tag{4}$$

$$\omega \cdot \psi_{n+1}(t) = \psi_n(\omega^2 t), \quad \omega \cdot \psi(t) = \psi(\omega^2 t) \qquad (0 \leqq t \leqq \omega^{-2}). \tag{5}$$

The curves Y_{n+1}, Y lie within the rectangle $(0, 0)$ $(\omega^{-1}, 0)$ $(\omega^{-1}, 1)$ $(0, 1)$ just as do the curves X_n, X within the rectangle $(0, 0)$ $(1, 0)$ $(1, 1)$ $(0, 1)$. Hence

$$\psi_{n+1}(t) = \varphi_n(\omega t), \quad \psi(t) = \varphi(\omega t) \qquad (0 \leqq t \leqq \omega^{-1}). \tag{6}$$

Similarly the curves X_n, X lie within the rectangle $(0, 0)$ $(\omega^{-1}, 0)$ $(\omega^{-1}, \omega^{-1})$ $(0, \omega^{-1})$ just as do the curves Y_n, Y within the rectangle $(0, 0)$ $(1, 0)$ $(1, 1)$ $(0, 1)$, and so

$$\omega \cdot \varphi_n(t) = \psi_n(\omega t), \quad \omega \cdot \varphi(t) = \psi(\omega t) \qquad (0 \leqq t \leqq \omega^{-1}). \tag{7}$$

Further from considerations of central symmetry in rectangles with center at $(\frac{1}{2}, \frac{1}{2})$ one has the relations (9–12) of I § 8, of which we need (10):

$$\varphi(t) + \varphi(1 - t) = 1, \quad \psi(t) + \psi(1 - t) = 1 \qquad (0 \leqq t \leqq 1). \tag{8}$$

6. *The complete curves X, Y; the continuous functions $\varphi(t)$, $\psi(t)$ of the*

unrestricted real argument t.– We have as definitions of $\varphi(t)$, $\psi(t)$ for $t>1$, $t<0$:

$$\varphi(t)=\omega^e\varphi(\omega^{-2e}t),\quad \psi(t)=\omega^e\psi(\omega^{-2e}t) \qquad (t>1) \tag{9}$$

where e is an (any) integer such that $\omega^{2e}\geqq t$,

$$\varphi(t)=-\varphi(-t),\quad \psi(t)=-\psi(-t) \qquad (t<0). \tag{10}$$

Similar definitions for the $\varphi_n(t)$, $\psi_n(t)$ being given, the relations 1, 2, 4, 5, 6, 7, 9, 10, hold for all values of t, t_1, t_2, e. One easily follows geometrically these extended curves as limit-curves. [These curves X, Y yield a curve C representing the complete t-line continuously upon the first and third xy-quadrants. Cf. I § 11.]

7. THEOREM.—The continuous curve Y has at no point V a tangent; the continuous function $y=\psi(t)$ has for no argument $t=t_0$ a derivative.– Instead of this* we consider the closer theorem of § 8. We desire results comparable (cf. § 11) with those of WEIERSTRASS for his continuous function without derivative.

8. THEOREM.†—The continuous curve Y with the equation

$$y=\psi(t)$$

has the following properties:

* The proof of this theorem is very immediate.— Setting

$$s(t,\ u)=\frac{\psi(t)-\psi(u)}{t-u}$$

we have for three arguments $t_1<t<t_2$ $s(t_1,\ t_2)$ intermediate in value between $s(t,\ t_1)$ and $s(t,\ t_2)$. This remark, apparent geometrically or from the formula

$$s(t_1,\ t_2)=\frac{t-t_1}{t_2-t_1}\,s(t,\ t_1)+\frac{t_2-t}{t_2-t_1}\,s(t,\ t_2),$$

due to THOMAE (DU BOIS REYMOND, Mathematische Annalen, vol. 16, p. 121), shows that if $\psi(t)$ has a derivative $\psi'(t)$ then $\psi'(t)$ is the limit necessarily existent of $s(t_1,\ t_2)$ on the set $t_1<t<t_2$ for $t_1=t$, $t_2=t$.— In our case one finds easily in every vicinity of (every) t *nodal* values t_1', t_2' $(t_1<t<t_2')$ such that $s(t_1',\ t_2')$ is numerically as large as may be wished and others t_1'', t_2'' $(t_1''<t<t_2'')$ such that $s(t_1'',\ t_2'')$ is zero or of opposite sign from $s(t_1',\ t_2')$, so that indeed $\psi'(t)$ exists for no t.

This proof is suggested by that of STEINITZ (loc. cit., p. 65).— Indeed our function $\psi(t)$ or $\psi(t;\ \omega)$ is (for $0\leqq t\leqq 1$) exactly Steinitz's function $f(t;\ \delta_1,\ \cdots,\ \delta_k,\ \cdots,\ \delta_{\omega^2})$ (l. c., p. 64) where $\delta_k=\delta_{(g-1)\omega+h}=(-1)^g\omega^{-1}$ $(g,\ h=1,\ 2,\ \cdots,\ \omega)$. It falls under the class of functions $f(t;\ \delta_1,\ \cdots,\ \delta_m)$ recognized by STEINITZ (l. c., p. 67, § 6) as having for no t a finite derivative, but not under his class of functions having for no t a finite or definitely infinite derivative (l. c., § 7). However if in defining this latter class (l. c., p. 68) one stipulates merely that every $\Delta_{1,\,t}$ shall be *either* of opposite sign from the corresponding $\Delta_{1,\,t}$ *or* zero one has a wider class possessing the desired property and including our functions $\psi(t;\ \omega)$.— The functions $f(t;\ \delta_1,\ \cdots,\ \delta_m)$ may easily be studied by the graphical methods of this paper.

† The parallel analytic statements as to the function $\psi(t)$ are omitted.

(α) Y has at no point V a progressive or a regressive non-vertical tangent;

(β) Y has at no point V both a progressive and a regressive vertical tangent;

(γ) Y has at and only at certain points $V = V(t)$ vertical progressive or regressive tangents:– the parameter t of such a point $V(t)$ being written in the scale of ω

$$t = \pm \omega^{z} t_0, \quad t_0 = \{. a_1 a_2 \cdots a_{2n-1} a_{2n} \cdots\} \quad \begin{pmatrix} z \text{ an (any) integer} \\ 0 \leqq t_0 < 1 \end{pmatrix},$$

there is no integer v such that for $i \gtrless v$ a_i is permanently $\omega - 1$ or permanently 0, but there is an integer u such that for $j \gtrless u$ either (γ_1) a_{2j-1} is permanently $\omega - 1$ and a_{2j} not permanently $\omega - 1$, or (γ_2) a_{2j-1} is permanently 0 and a_{2j} not permanently 0, and further μ_j denoting the (largest) number ($\geqq 0$) of the digits a_{2m} ($m = j+1, j+2, \cdots$) immediately succeeding a_{2j} and in value in the respective cases $\omega - 1$, 0, the relation

$$\lim_{j=\infty} (j - \mu_j) = +\infty$$

holds; then, $\eta(t)$ having the meaning $\eta_u(t_0)$, *i. e.*,

$$\eta(t) = (-1)^{\sum_{i=1}^{i=u} a_{2i-1}},$$

Y has at $V(t)$ a vertical tangent of slope $\eta(t) \cdot \infty$, which is (γ_1) progressive or regressive, (γ_2) regressive or progressive, according as t is positive or negative.

(δ) The points $V(t)$ of the curve Y separate into five sets: the sets of points at which Y has (δ_1) neither a progressive nor a regressive tangent; (δ_2, δ_3) no regressive but a progressive tangent of slope $+\infty$, $-\infty$; (δ_4, δ_5) no progressive but a regressive tangent of slope $+\infty$, $-\infty$. Each of these sets is everywhere dense on the curve Y.

The property (δ) follows from (α, β, γ) immediately. At the point $V(t)$ the directed secant $\overline{V(t)\ V(t_1)}$ of the curve Y has the *slope* $s(t, t_1)$

$$s(t, t_1) = \frac{\psi(t_1) - \psi(t)}{t_1 - t}. \tag{11}$$

In case on the set $t_1 > t$ the slope $s(t, t_1)$ (of the progressive secant) has for $t_1 = t$ a limit either finite or definitely infinite ($+\infty$, $-\infty$),

$$\lim_{\substack{t_1 > t \\ t_1 = t}} s(t, t_1) = \psi_p'(t), \tag{12}$$

then this limit $\psi_p'(t)$ is the slope of the progressive tangent at $V(t)$. The similar limit $\psi_r'(t)$ (if existent) of the slope of the regressive secant from $V(t)$ is the slope of the regressive tangent at $V(t)$.

The curve Y is odd, $\psi(-t) = -\psi(t)$; the regressive (progressive) secants at $V(t)$ $(t \leqq 0)$ have the same slopes as corresponding progressive (regressive) secants at $V(-t)$ $(-t \geqq 0)$.

The curve Y has the property $\psi(t) = \omega^e \psi(\omega^{-2e}t)$ (§ 6) and hence the relation

$$s(t,\ t_1) = \omega^{-e}\, s(\omega^{-2e}t,\ \omega^{-2e}t_1) \tag{13}$$

between the slopes of the corresponding progressive or regressive secants at $V(t)$, $V(\omega^{-2e}t)$ where $t \geqq 1$, $0 \leqq \omega^{-2e}t \leqq 1$.

But further the property $\psi(t) + \psi(1-t) = 1$ yields the relation

$$s(t,\ t_1) = s(1-t,\ 1-t_1) \qquad (0 < t < 1) \tag{14}$$

between the slopes of the regressive secants at $V(t)$ $(0 < t < 1)$ and those of the corresponding progressive secants at $V(1-t)$.

We see then easily that the theorem follows from that part of it relating to progressive tangents at points $V(t)$ $(0 \leqq t < 1)$.

9. *The proof* of the theorem* (§ 8) *as to progressive tangents to the curve Y at points $V(t)$* $(0 \leqq t < 1)$. – The argument t $(0 \leqq t < 1)$ written in the scale of ω (I § 10),

$$t = \{\,.\, a_1 a_2 \cdots a_{2n-1}\, a_{2n} \cdots\}, \tag{15}$$

is *nodal*, if ultimately every a_i is zero; otherwise it is say *ordinary*. The points $V(t)$ of the curve Y are correspondingly nodes or ordinary points.

The theorem for nodes. – The (general) node $V(t)$

$$t = \{\,.\, a_1 a_2 \cdots a_{2m-1} a_{2m}\} \qquad (a_{2m-1}a_{2m}) \neq (00)$$

appears as a node on the broken line Y_m; on $Y_n (n > m)$ the progressive link at $V(t)$, a progressive secant of Y, has the slope $s(t,\ t + \omega^{-2n}) = \eta_n(t) \cdot \omega^n$ (§ 3) while the secant-slope $s(t,\ t + 2\omega\omega^{-2n}) = 0$. Thus, *at no node has the curve Y a progressive tangent.*

The theorem for ordinary points. – The (general) ordinary point $V(t)$ has the argument t,

$$t = \{\,.\, a_1 a_2 \cdots a_{2m-1} a_{2m} \cdots\},$$

where there is no positive integer v such that either for every i $(i > v)$ $a_i = 0$ or for every i $(i > v)$ $a_i = \omega - 1$. We consider separately certain two supplementary classes of ordinary points.

(Class A). There is no integer u such that for every j $(j \geqq u)$ $a_{2j-1} = \omega - 1$,

* The reader is requested to have in vision the graphs suggested by the analytic phrasings of the text.

that is, there is an infinite sequence $j_1, j_2, \cdots, j_l, \cdots$ of increasing positive integers j_l such that a_{2j_l-1} has one of the values $0, 1, \cdots, \omega-2$. – We say that a point $V(t)$ of Y corresponds to the n-link of Y_n on which its corresponding point $V_n(t)$ lies. An ordinary point corresponds thus to a definite n-link for every n. Then for every l our point $V(t)$ corresponds to one of the ω j_l-links progressively sequent to the node $V(t_l)$:

$$t_l = \{\, .\, a_1 a_2 \cdots a_{2j_l-1} 0\,\}.$$

These ω links are the diagonals of ω rectangles R_{j_l} and together form the diagonal of a rectangle $R^{(l)}$ of dimensions $\omega\cdot\omega^{-2j_l} \times \omega\cdot\omega^{-j_l}$; the nodal extremities of this diagonal are $V(t_l)$, $V(t_l')$; the next ω_{j_l}-links form the diagonal of a rectangle $R^{[l]}$ of the same dimensions; its nodal extremities are $V(t_l')$, $V(t_l'')$. Here we have

$$\begin{aligned} t_l' &= t_l + \omega\omega^{-2j_l}, & \psi(t_l') &= \psi(t_l) + \eta^{(l)}\omega\omega^{-j_l}, & \eta^{(l)} &= \eta_{j_l}(t_l), \\ t_l'' &= t_l' + \omega\omega^{-2j_l}, & \psi(t_l'') &= \psi(t_l), & \eta^{[l]} &= -\eta^{(l)}. \end{aligned}$$

Setting

$$t_l' - t = \theta_l\,\omega\omega^{-2j_l}, \qquad \psi(t_l') - \psi(t) = \theta_l'\eta^{(l)}\omega\omega^{-j_l},$$

where θ_l, θ_l' are certain two numbers such that

$$0 < \theta_l < 1, \qquad 0 \leqq \theta_l' \leqq 1,$$

we find for the progressive secant-slopes $s(t, t_l')$, $s(t, t_l'')$ the values

$$s(t, t_l') = \eta^{(l)}\omega^{j_l}\theta_l'/\theta_l, \qquad s(t, t_l'') = -\eta^{(l)}\omega^{j_l}(1-\theta_l')/(1+\theta_l).$$

These slopes to two certain points in every progressive vicinity of $V(t)$ are of opposite sign (or one may vanish); further their difference taken numerically increases indefinitely with l. Thus, *at no ordinary point of class A has the curve Y a progressive tangent.*

(Class B). There is an integer u such that for every j $(j \geqq u)$ $a_{2j-1} = \omega - 1$. We have in this case (necessarily) an infinite sequence $j_1, j_2, \cdots, j_l, \cdots$ of increasing integers embracing of the integers $j > u$ all and only those for which $a_{2j} \neq \omega - 1$. For every j $(j > u)$ we denote by μ_j the (largest) number $(\geqq 0)$ of the digits a_{2m} $(m = j+1, j+2, j+3, \cdots)$ which immediately succeed a_{2j} and are each $\omega - 1$; thus $\mu_{j_l} = j_{l+1} - j_l - 1$. For the j's $(j \geqq u)$ the point $V(t)$ corresponds to j-links of progressive slopes $\eta_j(t)\omega^j = \eta_u(t)\omega^j$ all of the same sign $\eta_u(t)$. By consideration of the slopes to certain points in every progressive

vicinity of $V(t)$ we find on the sequence $\{\mu_j\}$ a necessary condition that the curve Y have at $V(t)$ a progressive tangent (necessarily vertical of slope $\eta_u(t)\cdot\infty$), and then we prove that this condition is sufficient.

We set and have

$$\eta = \eta_u(t), \qquad t_l = \{\,.\,a_1a_2\cdots a_{2j_l-1}a_{2j_l}\} \qquad (a_{2j_l}\leqq\omega-2),$$
$$t_l' = t_l + \omega^{-2j_l}, \qquad \psi(t_l') = \psi(t_l) + \eta\omega^{-j_l},$$
$$t_l'' = t_l' + \omega^{-2j_l}, \qquad \psi(t_l'') = \psi(t_l') + \eta\omega^{-j_l},$$
$$t_l''' = t_l' + \omega^{-2j_l}(1-\omega^{-1}), \qquad \psi(t_l''') = \psi(t_l'),$$
$$t_l' = t + \theta_l\omega^{-2j_l}\omega^{-2\mu_{j_l}-1}, \qquad \psi(t_l') = \psi(t) + \theta_l'\eta\omega^{-j_l}\omega^{-\mu_{j_l}},$$

where θ_l, θ_l' are certain two numbers such that

$$0<\theta_l<1, \quad \omega^{-1}\leqq\theta_l'\leqq 1.$$

We have then

$$s(t,\ t_l'') = \eta\omega^{j_l}(1+\theta_l'\omega^{-\mu_j})/(1+\theta_l\omega^{-2\mu_{j_l}-1}),$$
$$s(t,\ t_l''') = \eta\omega^{j_l-\mu_{j_l}}\theta_l'/(1-\omega^{-1}+\theta_l\omega^{-2\mu_{j_l}-1}).$$

From the form of $s(t,\ t_l'')$ one sees that *if the curve Y has at the ordinary point $V(t)$ of class B a progressive tangent its slope is $\eta\cdot\infty = \eta_u(t)\cdot\infty$, and* from the form of $s(t,\ t_l''')$ that *the μ_j have the property*

$$\overline{\mathrm{L}}_{j=\infty}(j-\mu_j) = +\infty, \tag{16}$$

since $j-\mu_j\geqq j_l-\mu_{j_l}$ for all j's for which $j_l\leqq j<j_{l+1}$.

Conversely, *the curve Y has at the ordinary point $V(t)$ of class B a progressive tangent of slope $\eta_u(t)\cdot\infty = \eta\cdot\infty$ if the infinite sequence $\{\mu_j\}$ determined by t has the property (16).* We take the general argument t' where $t<t'<t+\omega^{-2u}$ in the form

$$t' = \{\,.\,a_1'a_2'\cdots a_{2m-1}'a_{2m}'\cdots\}$$

and have for a certain integer $l(t')$ or say l $(j_l>u,\ l>1)$

$$a_i' = a_i, \quad a_{2j_l}'>a_{2j_l} \qquad (i=1,\ 2,\ \ldots,\ 2j_l-1),$$

so that indeed in terms of the earlier notation t_l' (with respect however to this particular l)

$$t' = t'_i + \theta(\omega - a_{2j_i} - 1)\omega^{-2j_i}, \quad \psi(t') = \psi(t'_i) + \theta'\eta(\omega - a_{2j_i} - 1)\omega^{-j_i},$$

where θ, θ' are certain two numbers depending each on t and t' such that

$$0 \leqq \theta < 1, \ 0 \leqq \theta' \leqq 1.$$

We have then

$$s(t, t') = \eta\omega^{j_i - \mu_{j_i}}\{\theta'_i + \theta'(\omega - a_{2j_i} - 1)\omega^{\mu_{j_i}}\}/\{\theta_i\omega^{-2\mu j_i - 1} + \theta(\omega - a_{2j_i} - 1)\},$$

and further, since $\theta'_i \geqq \omega^{-1}$, $\theta_i > 0$, $\theta \geqq 0$, $\theta' \geqq 0$, and by hypothesis

$$\underset{i=\infty}{\mathbf{L}}(j_i - \mu_{j_i}) = +\infty,$$

we have the desired conclusion

$$\underset{\substack{t' > t \\ t' = t}}{\mathbf{L}} s(t, t') = \eta \ \infty = \eta_u(t) \cdot \infty.$$

The conclusions thus reached as to progressive tangents to the curve Y at points $V(t)\,(0 \leqq t < 1)$ are those affirmed by the theorem of § 8. Hence that theorem is now fully proved.

10. *The curve X.* – By the use of the relation $\varphi(t) = \omega^{-1}\psi(\omega t)$ (§ 5) one easily translates the theorem of § 8 concerning the curve Y into an analogous one concerning the curve X.

11. *Comparison of tangential-properties of the curve Y and the Weierstrass curve W.* – WEIERSTRASS first exhibited (Cf. Crelle's Journal, vol. 79, p. 29, 1875; *Abhandlungen aus der Functionenlehre*, p. 97) a continuous function of the real variable having no derivative, viz.

$$f(t) = \sum_{n=0}^{\infty} b^n \cos(a^n t\pi),$$

where a is an odd integer, b is a positive number less than 1, and ab is greater than $1 + \frac{3}{2}\pi$. He proved that the yt-curve W, $y = f(t)$, has in the immediate vicinity of every point secants progressive and regressive of slope numerically as large as one will and of opposite signs. The curve W has then at no point a tangent and at no point a progressive or a regressive non-vertical tangent. So far as I know it has not been determined that at no point or at what points the curve W has (1) a progressive vertical tangent, (2) a regressive vertical tangent, (3) a progressive vertical and a regressive vertical tangent of oppo-

site slope (cusp with vertical tangent). – We have thus in these regards a closer knowledge of our curve Y, and this is true in comparison also with the other continuous tangentless curves which have been exhibited.

12. The curve Y has within every interval for every given positive number G points of arguments t_1, t, t_2 $(t_1 < t < t_2)$ such that $s(t, t_1)$, $s(t, t_2)$ are of opposite sign and numerically larger than G, for example, three successive nodes of Y_n, the n and the nodes being properly chosen (§ 3). Hence,* within every interval for every constant C finite or definitely infinite a point $V(t)$ exists such that in every vicinity of $V(t)$ the slopes $s(t, t_1)$ are dense at C.

13. We inquire whether there exists a zt-curve of the equation $z = \chi(t)$, where $\chi(t)$ is a single-valued continuous function of the real variable t, such that in every vicinity of every point (z, t) of the curve both the progressive and the regressive slopes $s(t, t_1)$ are dense at $+\infty$ and at $-\infty$. *No such curve exists*, since on every interval i every continuous function $\chi(t)$ assumes for some value $t = t'$ of the interval a maximum value and then at the point (z', t') of the curve $z = \chi(t)$ the secant-slopes $s(t', t_1)$ to points (z_1, t_1) of argument t_1 of the interval i, if progressive, are all negative or zero, and, if regressive, are all positive or zero.

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* By an obvious generalization of a theorem due to KÖNIG (Cf. p. 12 of the memoir, *Über stetige Functionen, die innerhalb jedes Intervalls extreme Werte besitzen*, Monatshefte für Mathematik und Physik, vol. 1, pp. 7–12, 1890).

A NEW DEFINITION OF THE GENERAL ABELIAN LINEAR GROUP*

BY

LEONARD EUGENE DICKSON

1. We may give a striking definition of the general Abelian group, making use of the fruitful conception of the "compounds of a given linear homogeneous group," introduced in recent papers by the writer.† In § 3 we determine the multiplicity of the isomorphism of a given linear homogeneous group to its compound groups. This result is applied in § 4 to show the simple relation of the Abelian group to the general linear homogeneous group in the same number of variables. In § 5 it is shown that the simple groups of composite order which are derived from the decompositions of the quaternary Abelian group and the quinary orthogonal group, each in the $GF[p^n]$, $p > 2$, are simply isomorphic. The investigation affords a proof of the simple isomorphism between the corresponding ten-parameter projective groups without the consideration of their infinitesimal transformations.

2. It will be convenient to introduce a notation more compact than that usually employed‡ for the substitutions of the general Abelian group $A_{2m,\,p^n}$ on $2m$

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† *Concerning a linear homogeneous group in $C_{m,\,q}$ variables isomorphic to the general linear homogeneous group in m variables*, Bulletin, Dec., 1898.

The structure of certain linear groups with quadratic invariants, Proceedings of the London Mathematical Society, vol. 30, pp. 70–98.

‡ Dickson, *The Structure of the Hypoabelian Groups*, Bulletin, pp. 495–510, July, 1898; *A Triply Infinite System of Simple Groups*, The Quarterly Journal of Pure and Applied Mathematics, pp. 169–178, 1897; Jordan, *Traité des Substitutions*, pp. 171–179, for the case $n = 1$.

Instead of considering the cogredient linear substitutions leaving invariant (up to a factor a) the usual bilinear function it is convenient to consider here the substitutions A leaving invariant the function

$$\psi \equiv \sum_{j=1}^{m} \begin{vmatrix} \xi_{2j-1} & \xi_{2j} \\ \xi_{2j-1}^{p^n} & \xi_{2j}^{p^n} \end{vmatrix}$$

The conditions that A shall leave ψ invariant are seen to be (1). The *hyperabelian* group of linear homogeneous substitutions in the $GF[p^{2n}]$ on $2m$ indices which leave ψ invariant has been studied by the writer in an article presented to the London Mathematical Society.

indices in the Galois field of order p^n. The conditions that a substitution

$$A: \qquad \xi'_i = \sum_{j=1}^{2m} a_{ij}\xi_j \qquad (i=1, \cdots, 2m)$$

shall be Abelian are the following:

$$(1) \qquad \sum_{j=1}^{m} \begin{vmatrix} a_{2j-1\,i} & a_{2j-1\,k} \\ a_{2j\,i} & a_{2j\,k} \end{vmatrix} = a\varepsilon_{ik} \qquad (i, k=1, \cdots, 2m;\ i<k)$$

where a is a parameter $\neq 0$ depending upon the particular substitution A and where every $\varepsilon_{ik}=0$ unless $k=i+1=$ even, when

$$\varepsilon_{2l-1\,2l}=1 \qquad (l=1, \cdots, m).$$

The second compound $C_{2m,2}$ of the $2m$-ary group A_{2m,p^n} is formed by the substitutions

$$(2) \qquad Y'_{i_1 i_2} = \sum_{j_1<j_2}^{j_1, j_2 = 1 \ldots 2m} \begin{vmatrix} a_{i_1 j_1} & a_{i_1 j_2} \\ a_{i_2 j_1} & a_{i_2 j_2} \end{vmatrix} Y_{j_1 j_2} \qquad \left(\begin{matrix} i_1, i_2=1, \cdots, 2m \\ i_1<i_2 \end{matrix}\right).$$

We readily verify that the group $C_{2m,2}$ has the relative invariant

$$Z \equiv \sum_{i=1}^{m} Y_{2i-1\,2i}.$$

Indeed, in virtue of the relations (1), we have, on applying to Z the substitution (2),

$$\sum_{i=1}^{m} Y'_{2i-1\,2i} = \sum_{j_1, j_2} \left\{ \sum_{i=1}^{m} \begin{vmatrix} a_{2i-1\,j_1} & a_{2i-1\,j_2} \\ a_{2i\,j_1} & a_{2i\,j_2} \end{vmatrix} \right\} Y_{j_1 j_2}$$

$$= a \sum_{j_1, j_2} \varepsilon_{j_1 j_2} Y_{j_1 j_2} = a \sum_{i=1}^{m} Y_{2i-1\,2i}.$$

Inversely, if the substitution (2) multiply the function Z by a constant a, the relations (1) hold true. We have proved the result:

THEOREM.—*The general Abelian group A_{2m,p^n} is the largest $2m$-ary linear homogeneous group whose second compound has as a relative invariant the linear function Z.*

3. To establish the more important theorem of § 4, we determine the multiplicity of the isomorphism of a given m-ary linear homogeneous group G_m to its q^{th} compound $C_{m,q}$, supposing that $q<m$. To the substitution

$$(a_{ij}) \qquad (i, j=1, \cdots, m)$$

of G_m there corresponds the following substitution of $C_{m,q}$: *

* We employ Sylvester's umbral notation for determinants.

$$[a]_q: \qquad Y'_{i_1 \cdots i_q} = \sum_{\substack{j_1 \cdots j_q \\ j_1 < j_2 \cdots < j_q}}^{1 \cdots m} \begin{vmatrix} i_1 \cdots i_q \\ j_1 \cdots j_q \end{vmatrix} Y_{j_1 \cdots j_q} \qquad \left(\begin{matrix} i_1, \cdots, i_q = 1, \cdots, m \\ i_1 < i_2 < \cdots < i_q \end{matrix} \right).$$

Let j be an integer such that $q < j \leqq m$. Consider the matrix J of certain coefficients of the substitution $[a]_q$, viz.,

$$\begin{pmatrix} \begin{vmatrix} 1\,2 \cdots q-1\,q \\ 1\,2 \cdots q-1\,q \end{vmatrix} & -\begin{vmatrix} 1\,2 \cdots q-1\,q \\ 1\,2 \cdots q-1\,j \end{vmatrix} \cdots (-1)^q \begin{vmatrix} 1\,2 \cdots q-1\,q \\ 2\,3 \cdots q \quad j \end{vmatrix} \\ -\begin{vmatrix} 1\,2 \cdots q-1\,j \\ 1\,2 \cdots q-1\,q \end{vmatrix} & \begin{vmatrix} 1\,2 \cdots q-1\,j \\ 1\,2 \cdots q-1\,j \end{vmatrix} \cdots (-1)^{q+1} \begin{vmatrix} 1\,2 \cdots q-1\,j \\ 2\,3 \cdots q \quad j \end{vmatrix} \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots \\ (-1)^q \begin{vmatrix} 2\,3 \cdots q \quad j \\ 1\,2 \cdots q-1\,q \end{vmatrix} & (-1)^{q+1} \begin{vmatrix} 2\,3 \cdots q \quad j \\ 1\,2 \cdots q-1\,j \end{vmatrix} \cdots & \begin{vmatrix} 2\,3 \cdots q \quad j \\ 2\,3 \cdots q \quad j \end{vmatrix} \end{pmatrix}.$$

Consider also the matrix A of determinant Δ,

$$A \equiv \begin{pmatrix} a_{jj}\, a_{qj} \cdots a_{1j} \\ a_{jq}\, a_{qq} \cdots a_{1q} \\ \cdots\cdots\cdots\cdots \\ a_{j1}\, a_{q1} \cdots a_{11} \end{pmatrix}.$$

The composition of the matrices J and A gives the result

$$JA \equiv \begin{pmatrix} \Delta\, 0 \cdots 0 \\ 0\, \Delta \cdots 0 \\ \cdots\cdots\cdots \\ 0\, 0 \cdots \Delta \end{pmatrix}.$$

We seek those substitutions of G_m which correspond to the identity in $C_{m,q}$. Suppose, therefore, that $[a]_q$ reduces to the identical substitution, so that the matrix J is the identity.

In this case we have

$$\Delta^{q+1} = \Delta, \; a_{ik} = 0, \; a_{ii} = \Delta \qquad (i, k = 1, 2 \cdots, q, j; \; i \neq k).$$

Taking in turn $j = q+1, q+2, \cdots, m$, we have the result

$$(a_{ij}) \equiv \begin{pmatrix} \Delta\, 0 \cdots 0 \\ 0\, \Delta \cdots 0 \\ \cdots\cdots\cdots \\ 0\, 0 \cdots \Delta \end{pmatrix}.$$

Hence $\Delta \neq 0$ and therefore $\Delta^q = 1$. Inversely, every such substitution of G_m corresponds to the identity in $C_{m,q}$.

THEOREM.—*The continuous group of all m-ary linear homogeneous substitutions is* $(q, 1)$ *fold isomorphic to its* q^{th} *compound* $(q < m)$.

For linear substitutions in the $GF[p^n]$, we have

$$\Delta^q = 1, \quad \Delta^{p^n-1} = 1.$$

Thus we have the analogous.

THEOREM.—*The group of all m-ary linear homogeneous substitutions in the* $GF[p^n]$ *is* $(g, 1)$ *fold isomorphic to its* q^{th} *compound,* g *being the greatest common divisor of* q *and* $p^n - 1$.

4. From the results of §§ 2–3, we derive immediately the

THEOREM.—*According as* $p = 2$ *or* $p > 2$, *the general Abelian group* A_{2m,p^n} *is holoedrically or hemiedrically* isomorphic to that subgroup of the second compound of the general* $2m$*-ary linear homogeneous group in the* $GF[p^n]$ *which has as a relative invariant the linear function* Z.

The writer has shown (Bulletin, l. c.) that this second compound leaves invariant the Pfaffian

$$F_{2m} \equiv [1, 2, \cdots, 2m].$$

Hence A_{2m,p^n} is isomorphic to a linear homogeneous group in $m(2m-1)$ variables Y_{ij} with coefficients in the $GF[p^n]$ and having as relative invariants the functions

$$F_{2m}, \quad Z \equiv \sum_{i=1}^{m} Y_{2i-1\,2i}.$$

To the subgroup* of the latter which leaves these functions absolutely invariant there corresponds a self-conjugate subgroup of A_{2m,p^n}, which leaves the customary bilinear function absolutely invariant. This subgroup, containing only substitutions of determinant ± 1, may be designated as the *special* Abelian group A'_{2m,p^n}. It has the self-conjugate substitution which changes the signs of all the $2m$ indices. Except for $(2m, p^n) = (2, 2), (2, 3)$ and $(4, 2)$, the quotient group H_{2m,p^n} is simple.‡

5. THEOREM.—*For* $p > 2$, *the simple group* H_{4,p^n}, *having the order*

* Since the Abelian group contains the substitution $\xi_i' = -\xi_i (i = 1, 2, \cdots, 2m)$.

† This group has been studied by the writer in the Proceedings of the London Mathematical Society, l. c., §§ 22–33.

‡ DICKSON, *A triply infinite system of simple groups*, The Quarterly Journal of Mathematics, July, 1897; *ibid.*, April, 1899, for the cases $p = 2$, $2m = 4$, $n > 1$, previously unconsidered.

$$\tfrac{1}{2}(p^{4n}-1)(p^{2n}-1)p^{4n},$$

is simply isomorphic to the simple subgroup of equal order of the quinary orthogonal group in the $GF[p^n]$.

On introducing the invariant $Z \equiv Y_{12} + Y_{34}$ as a new variable in place of Y_{34}, the general substitution [see (2)] of the second compound of A'_{4,p^n} becomes, for $p > 2$:

	$Y_{12}-\frac{1}{2}Z$	Y_{13}	Y_{14}	Y_{23}	Y_{24}
$(Y_{12}-\frac{1}{2}Z)' =$	$2\begin{vmatrix}1&2\\1&2\end{vmatrix}-1$	$\begin{vmatrix}1&2\\1&3\end{vmatrix}$	$\begin{vmatrix}1&2\\1&4\end{vmatrix}$	$\begin{vmatrix}1&2\\2&3\end{vmatrix}$	$\begin{vmatrix}1&2\\2&4\end{vmatrix}$
$Y_{13}' =$	$2\begin{vmatrix}1&3\\1&2\end{vmatrix}$	$\begin{vmatrix}1&3\\1&3\end{vmatrix}$	$\begin{vmatrix}1&3\\1&4\end{vmatrix}$	$\begin{vmatrix}1&3\\2&3\end{vmatrix}$	$\begin{vmatrix}1&3\\2&4\end{vmatrix}$
$Y_{14}' =$	$2\begin{vmatrix}1&4\\1&2\end{vmatrix}$	$\begin{vmatrix}1&4\\1&3\end{vmatrix}$	$\begin{vmatrix}1&4\\1&4\end{vmatrix}$	$\begin{vmatrix}1&4\\2&3\end{vmatrix}$	$\begin{vmatrix}1&4\\2&4\end{vmatrix}$
$Y_{23}' =$	$2\begin{vmatrix}2&3\\1&2\end{vmatrix}$	$\begin{vmatrix}2&3\\1&3\end{vmatrix}$	$\begin{vmatrix}2&3\\1&4\end{vmatrix}$	$\begin{vmatrix}2&3\\2&3\end{vmatrix}$	$\begin{vmatrix}2&3\\2&4\end{vmatrix}$
$Y_{24}' =$	$2\begin{vmatrix}2&4\\1&2\end{vmatrix}$	$\begin{vmatrix}2&4\\1&3\end{vmatrix}$	$\begin{vmatrix}2&4\\1&4\end{vmatrix}$	$\begin{vmatrix}2&4\\2&3\end{vmatrix}$	$\begin{vmatrix}2&4\\2&4\end{vmatrix}$

It is therefore a substitution on five indices leaving absolutely invariant the function

$$\varphi \equiv (\tfrac{1}{2}Z)^2 - [1234] \equiv (Y_{12}-\tfrac{1}{2}Z)^2 + Y_{13}Y_{24} - Y_{14}Y_{23}.$$

This second compound is simply isomorphic to H_{4,p^n}. Indeed, the former is hemiedrically isomorphic to A'_{4,p^n} by §3; while to the substitution changing the sign of every index there corresponds the identity in the second compound.

By a simple transformation of indices* the function φ can be given the form

$$\sum_{i=1}^{5} x_i^2.$$

Hence the second compound is simply isomorphic to a subgroup O_{5,p^n} of the total orthogonal group O of determinant unity. From the result of §16 of the paper in the Proceedings of the London Mathematical Society, above cited, it follows that O_{5,p^n} does not contain the substitution which extends the simple subgroup of O of order

*See the first pages of the article, *Determination of the structure of all linear homogeneous groups in a Galois field which are defined by a quadratic invariant,* American Journal of Mathematics, July, 1899.

$$\tfrac{1}{2}(p^{4n}-1)(p^{2n}-1)p^{4n}$$

to the total group O. Hence O_{5,p^n} is this simple subgroup.

This investigation also proves the theorem due to Lie: *The projective group of a linear complex in space of three dimensions is isomorphic to the projective group of a non-degenerate surface of the second order in space of four dimensions, each group having ten parameters.*

UNIVERSITY OF CALIFORNIA, *August* 20, 1898.